PhaseLocked Systems

Technical Report PHS0251

Probability Distributions

Robert T. Short
PhaseLocked Systems
rtshort@ieee.org

Version 1.1
Apr 27 2012

Abstract

This is a quick-reference guide to a few commonly used probability distributions. The key properties of several distributions are tabulated and the text contains brief derivations.

Forward

I am continually looking up moments, moment generating functions, and other properties of random variables in one textbook or another. The problem is that normal random variables are in this book, chi-squared in that book, and lognormal in some other book and I can't remember which book until I find it. I got annoyed and built this little summary report. As I need additional distributions I add them to this report.

This document may be found at `http://www.phaselockedsystems.com/publications`. Comments and corrections are welcome and may be sent to the email address on the cover page.

1 Introduction and Summary

This is a quick reference guide to a few useful random variables. It is not a textbook and no attempt is made at explanation. Tables 1 through 9 contain summaries of the properties of these random variables. Section 2 is a brief description of the notation used. The remainder of the document derives the properties in the following tables.

None of the information found in this report is new or unusual, but some of the details are not normally found in basic textbooks. In particular Craig's Q function, the sections on noncentral chi-squared and Rice random variables, and the section on lognormal random variables took a little digging and deriving to construct. Otherwise this stuff is found in readily available references, and there are about an Avogadro's number of such textbooks. My primary references are normally [1], [2] and [3]. Other good sources are [4], [5], and [6].

Table 1: Properties of Uniform Random Variables

Property	Details
Notation	$\mathcal{U}(a,b)$
Density	$f_{\mathcal{U}}(x;a,b) = \frac{1}{b-a}\chi_{[a,b]}(x)$
mean	$\frac{b+a}{2}$
variance	$\frac{(b-a)^2}{12}$
k^{th} moment	$\frac{1}{k+1}\frac{b^{k+1}-a^{k+1}}{b-a}$
mgf	$\frac{e^{sb}-e^{sa}}{(b-a)s}$
Q function	$\mathrm{Q}_{\mathcal{U}}(x) = \begin{cases} 0 & x < 0 \\ x & x \in [0,1] \\ 1 & x > 1 \end{cases}$
	$P\{\mathcal{U}(a,b) > y\} = \mathrm{Q}_{\mathcal{U}}\left(\frac{y-a}{b-a}\right)$

Table 2: Properties of Normal Random Variables

Property	Details
Notation	$\mathcal{N}(\mu,\sigma^2)$
Density	$f_{\mathcal{N}}(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
mean	μ
variance	σ^2
k^{th} central moment (k odd)	0
k^{th} central moment (k even)	$1\cdot 3\cdot\ldots\cdot(k-1)\sigma^k = \sqrt{\frac{2^k\sigma^{2k}}{\pi}}\Gamma\left(\frac{k+1}{2}\right)$
mgf	$e^{\mu s+\frac{\sigma^2 s^2}{2}}$
replication	$\mathbf{x}_k \sim \mathcal{N}(\mu_k,\sigma_k^2)$ independent $\Rightarrow \mathbf{x}_1+\mathbf{x}_2 \sim \mathcal{N}(\mu_1+\mu_2,\sigma_1^2+\sigma_2^2)$
Q function	$\mathrm{Q}_{\mathcal{N}}(x) = \mathrm{Q}(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-u^2/2}du$
	$P\{\mathcal{N}(\mu,\sigma^2) > y\} = \mathrm{Q}\left(\frac{y-\mu}{\sigma}\right)$

Table 3: Properties of Gamma Random Variables

Property	Details
Notation	$\Gamma(\nu,\beta)$
Density	$f_\Gamma(x;\nu,\beta) = \frac{1}{\beta^\nu\Gamma(\nu)} x^{\nu-1} e^{-x/\beta}\chi_{[0,\infty)}(x)$
mean	$\nu\beta$
variance	$\nu\beta^2$
k^{th} moment	$\nu(\nu+1)(\nu+2)\ldots(\nu+k-1)\beta^k = \frac{\Gamma(\nu+k)}{\Gamma(\nu)}\beta^k$
mgf	$\frac{1}{(1-\beta s)^\nu}$ for $\beta s < 1$
replication	$\mathbf{x}_k \sim \Gamma(\nu_k,\beta)$ independent then $\mathbf{x}_1+\mathbf{x}_2 \sim \Gamma(\nu_1+\nu_2,\beta)$
Q function	$\mathrm{Q}_\Gamma(x;\nu) = \frac{1}{\Gamma(\nu)}\int_x^\infty u^{\nu-1}e^{-u}du = \Gamma(\nu,x)$
	$P\{\Gamma(\nu,\beta) > y\} = \mathrm{Q}_\Gamma(y/\beta;\nu)$

Table 4: Properties of Exponential Random Variables

Property	Details
Notation	$\mathcal{E}(\beta)$
	$\sim \Gamma(1,\beta)$
Density	$f_\mathcal{E}(x;\beta) = \frac{1}{\beta}e^{-x/\beta}\chi_{[0,\infty)}(x)$
mean	β
variance	β^2
k^{th} moment	$k!\beta^k$
mgf	$\frac{1}{1-\beta s}$
Q function	$\mathrm{Q}_\mathcal{E}(x) = e^{-x}$
	$P\{\mathcal{E}(\beta) > y\} = e^{-y/\beta}$

Table 5: Properties of χ^2 Random Variables

Property	Details
Notation	$\chi^2(n,\sigma^2)$
Definition	$\mathbf{y} = \sum_{k=1}^n \mathbf{x_k}^2$ with $\mathbf{x}_k \sim \mathcal{N}(0,\sigma^2)$ independant
	$\sim \Gamma(n/2, 2\sigma^2)$
Density	$f_{\chi^2}(y;n,\sigma^2) = \frac{1}{(2\sigma^2)^{n/2}\Gamma(n/2)} y^{n/2-1}e^{-y/2\sigma^2}\chi_{[0,\infty)}(y)$
mean	$n\sigma^2$
variance	$2n\sigma^4$
k^{th} moment	$\frac{\Gamma(n/2+k)}{\Gamma(n/2)}(2\sigma^2)^k$
mgf	$\frac{1}{(1-2\sigma^2 s)^{n/2}}$
Q function	$\mathrm{Q}_{\chi^2}(x;n) = \frac{1}{\Gamma(n/2)}\int_x^\infty u^{n/2-1}e^{-u}du = \Gamma(n/2,x)$
n even	$\mathrm{Q}_{\chi^2}(x;n) = e^{-x}\sum_{k=0}^{n/2-1}\frac{x^k}{k!}$
	$P\{\chi^2(n,\sigma^2) > y\} = \mathrm{Q}_{\chi^2}(y/2\sigma^2;n)$

Table 6: Properties of Noncentral χ^2 Random Variables

Property	Details
Notation	$\bar{\chi}^2(n, \mu^2, \sigma^2)$
Definition	$\mathbf{y} = \sum_{k=1}^{n} \mathbf{x_k}^2$ with $\mathbf{x}_k \sim \mathcal{N}(\mu_k, \sigma^2)$ $\mu^2 = \sum_{k=1}^{n} \mu_k^2$
Density	$f_{\bar{\chi}^2}(y; n, \mu^2, \sigma^2) = \frac{1}{2\sigma^2}\left(\frac{y}{\mu^2}\right)^{(n-2)/4} e^{-(\mu^2+y)/2\sigma^2} \mathrm{I}_{n/2-1}\left(\frac{\sqrt{y\mu^2}}{\sigma^2}\right)\chi_{(0,\infty]}(y)$
mean	$n\sigma^2 + \mu^2$
variance	$2n\sigma^4 + 4\sigma^2\mu^2$
k^{th} moment	$(2\sigma^2)^k \frac{\Gamma(k+n/2)}{\Gamma(n/2)} {}_1\mathrm{F}_1(-k, n/2; -\mu^2/2\sigma^2)$
k^{th} moment (n even)	$(2\sigma^2)^k k! L_k^{(n/2-1)}(-\mu^2/2\sigma^2)$
mgf	$\frac{1}{(1-2s\sigma^2)^{n/2}} e^{\frac{\mu^2 s}{1-2s\sigma^2}}$
Q function	$\mathrm{Q}_{\bar{\chi}^2}(x; n, m^2) = \int_x^{\infty} (\sqrt{w})^{n/2-1} e^{-(m^2+w)} \frac{\mathrm{I}_{n/2-1}\left(2\sqrt{m^2 w}\right)}{m^{n/2-1}} dw$ $P\{\bar{\chi}^2(n, \mu^2, \sigma^2) > y\} = \mathrm{Q}_{\bar{\chi}^2}(y/2\sigma^2; n, \mu^2/2\sigma^2)$

Table 7: Properties of Rayleigh Random Variables

Property	Details
Notation	$\mathcal{R}(n, \sigma^2)$
Definition	$\mathbf{y} = \sqrt{\sum_{k=1}^{n} \mathbf{x_k}^2}$ with $\mathbf{x}_k \sim \mathcal{N}(0, \sigma^2)$
Density	$f_{\bar{\mathcal{R}}}(y; n, \sigma^2) = \frac{2}{(2\sigma^2)^{n/2}\Gamma(n/2)} y^{n-1} e^{-y^2/2\sigma^2} \chi_{[0,\infty)}$
k^{th} moment	$(2\sigma^2)^{k/2} \frac{\Gamma((n+k)/2)}{\Gamma(n/2)}$
Q function	$\mathrm{Q}_{\mathcal{R}}(x; n) = \frac{1}{2^{n/2-1}\Gamma(n/2)} \int_x^{\infty} u^{n-1} e^{-u^2/2} du = \Gamma(n/2, x^2/2)$ $P\{\mathcal{R}(n, \sigma^2) > y\} = \mathrm{Q}_{\mathcal{R}}(y/\sigma) = \Gamma(n/2, y^2/2\sigma^2)$

Table 8: Properties of Rice Random Variables

Property	Details
Notation	$\bar{\mathcal{R}}(n, \mu^2, \sigma^2)$
Definition	$\mathbf{y} = \sqrt{\sum_{k=1}^{n} \mathbf{x_k}^2}$ with $\mathbf{x}_k \sim \mathcal{N}(\mu_k, \sigma^2)$ $\mu^2 = \sum_{k=1}^{n} \mu_k^2$
Density	$f_{\bar{\mathcal{R}}}(y; n, \mu^2, \sigma^2) = \frac{1}{\sigma^2} y^{n/2} e^{-\frac{\mu^2+y^2}{2\sigma^2}} \frac{\mathrm{I}_{n/2-1}\left(\frac{\mu y}{\sigma^2}\right)}{\mu^{n/2-1}} \chi_{(0,\infty]}(y)$
k^{th} moment	$(2\sigma^2)^{k/2} e^{-\mu^2/2\sigma^2} \frac{\Gamma((n+k)/2)}{\Gamma(n/2)} {}_1\mathrm{F}_1\left(\frac{n+k}{2}, \frac{n}{2}; \frac{\mu^2}{2\sigma^2}\right)$
Q function	$\mathrm{Q}_{\bar{\mathcal{R}}}(x; n, m^2) = \mathrm{Q}_{n/2}(m, x) = \int_x^{\infty} u^{n/2} e^{-\frac{m^2+u^2}{2}} \frac{I_{n/2-1}(mu)}{m^{n/2-1}} du$ $P\{\bar{\mathcal{R}}(n, \mu^2, \sigma^2) > y\} = \mathrm{Q}_{n/2}\left(\frac{m}{\sigma}, \frac{y}{\sigma}\right)$

Table 9: Properties of the Lognormal Random Variable

Property	Details
Notation	$\mathcal{L}(\mu, \sigma^2, \alpha, \beta)$
Definition	$\mathbf{y} = \alpha^{\mathbf{x}/\beta}$ with $\mathbf{x} \sim \mathcal{N}(\mu, \sigma^2)$
Density	$f_{\mathcal{L}}(y; \mu, \sigma^2, \alpha, \beta) = \frac{\beta}{\log \alpha} \frac{1}{y\sqrt{2\pi\sigma^2}} e^{-(\beta \log_\alpha y - \mu)^2/2\sigma^2} \chi_{[0,\infty)}(y)$
k^{th} moment	$\tilde{\mu}^k e^{\frac{1}{2}\left[\log \tilde{\sigma}^k\right]^2}$
	$\tilde{\mu} = \alpha^{\mu/\beta}$, $\tilde{\sigma} = \alpha^{\sigma/\beta}$
Definition	$\mathbf{y} = e^{\mathbf{x}}$ with $\mathbf{x} \sim \mathcal{N}(\mu, \sigma^2)$
Density	$f_{\mathcal{L}}(y; \mu, \sigma^2, e, 1) = \frac{1}{y\sqrt{2\pi\sigma^2}} e^{-(\log y - \mu)^2/2\sigma^2} \chi_{[0,\infty)}(y)$
Mean	$e^{\mu + \sigma^2/2}$
Variance	$2e^{2\mu + 3\sigma^2/2} \sinh(\sigma^2)$
k^{th} Moment	$e^{k\mu + k^2\sigma^2/2}$
	$\tilde{\mu} = \mu$, $\tilde{\sigma} = \sigma$
Definition	$\mathbf{y} = 10^{\mathbf{x}/20}$ with $\mathbf{x} \sim \mathcal{N}(\mu, \sigma^2)$
Density	$f_{\mathcal{L}}(y; \mu, \sigma^2, 10, 20) = \frac{20}{\log 10} \frac{1}{y\sqrt{2\pi\sigma^2}} e^{-(20 \log_{10} y - \mu)^2/2\sigma^2} \chi_{[0,\infty)}(y)$
n^{th} Moment	$\tilde{\mu}^n e^{\frac{1}{2}[\log \tilde{\sigma}^n]^2}$
	$\tilde{\mu} = 10^{\mu/20}$, $\tilde{\sigma} = 10^{\sigma/20}$

2 Notation and Definitions

We will use boldface letters to represent random variables, e.g. $\mathbf{x}$. A random variable (rv) is described by a *density* or *probability density function* (pdf) $f_{\mathbf{x}}(x)$. Equally descriptive is the *distribution function* or *cumulative distribution function* (cdf)

$$F_{\mathbf{x}}(x) = P\{\mathbf{x} \leq x\} = \int_{-\infty}^{x} f_{\mathbf{x}}(u)du. \tag{1}$$

The complementary distribution function is

$$1 - F_{\mathbf{x}}(x) = \int_{x}^{\infty} f_{\mathbf{x}}(u)du. \tag{2}$$

In this report, we are most interested in tabulating the properties of commonly used random variables with well known densities. If a random variable $\mathbf{x}$ is distributed according to a particular class, say $\mathcal{D}$, we say that

$$\mathbf{x} \sim \mathcal{D} \tag{3}$$

or "The random variable $\mathbf{x}$ is distributed according to the properties of $\mathcal{D}$". For example, if $\mathbf{x}$ is normally distributed with mean μ and variance σ^2, we write $\mathbf{x} \sim \mathcal{N}(\mu, \sigma^2)$. We will write the density and distribution as $f_{\mathcal{D}}$ and $F_{\mathcal{D}}$ respectively. Writing

$$P\{\mathcal{D} \in S\} \tag{4}$$

means the probability that a random variable of type $\mathcal{D}$ is in the set S.

The k^{th} moment of a random variable $\mathbf{x}$ is

$$E\mathbf{x}^k = \int_{-\infty}^{\infty} u^k f_{\mathcal{D}}(u)du. \tag{5}$$

The mean, which we will designate $\mu_{\mathcal{D}}$, is the first moment. The k^{th} central moment is

$$E[\mathbf{x} - \mu_{\mathcal{D}}]^k = \int_{-\infty}^{\infty} [u - \mu_{\mathcal{D}}]^k f_{\mathcal{D}}(u)du. \tag{6}$$

The variance, written $\sigma^2_{\mathcal{D}}$, is the second central moment.

The moment generating function (mgf) is defined as[1]

$$\mathrm{M}_{\mathcal{D}}(s) = Ee^{s\mathbf{x}} = \int_{-\infty}^{\infty} e^{su} f_{\mathcal{D}}(u)du. \tag{7}$$

If $\mathrm{M}_{\mathcal{D}}$ is analytic is a region containing zero, then the moments may be computed using the formula

$$E\mathbf{x}^k = \frac{d^k}{ds^k}\mathrm{M}_{\mathcal{D}}(0) \tag{8}$$

and if we expand the mgf in a Taylor series, we have

$$\mathrm{M}_{\mathcal{D}}(s) = \sum_{k=0}^{\infty} E\mathbf{x}^k \frac{s^k}{k!}. \tag{9}$$

[1]Some authors use $Ee^{-s\mathbf{x}}$.

The mgf is essentially the Laplace transform of the density, and so if the mgf is known, the density is uniquely determined by inverting the mgf. Similarly the *characteristic function* (cf) is the Fourier transform (with the frequency variable sign-inverted) of the density, so

$$\Phi(\nu) = \mathrm{M}(i\nu) = Ee^{i\nu\mathbf{x}}. \tag{10}$$

If $\mathbf{x}$ and $\mathbf{y}$ are independant r.v.s and $\mathbf{z} = \mathbf{x} + \mathbf{y}$, then the density of $\mathbf{z}$ is the convolution of the densities of $\mathbf{x}$ and $\mathbf{y}$ and the mgf of $\mathbf{z}$ is the product of the mgf of $\mathbf{x}$ and $\mathbf{y}$. If $\mathbf{z} = g(\mathbf{x})$ the density of $\mathbf{z}$ may be found by solving $z = g(x)$ for x. If $\{x_n; n = 1 \dots N\}$ are all the solutions, the density of $\mathbf{z}$ is

$$f_{\mathbf{z}} = \sum_{n=1}^{N} \frac{f_{\mathbf{x}}(x_n)}{|g'(x_n)|}. \tag{11}$$

Furthermore

$$E\mathbf{z} = Eg(\mathbf{x}) = \int_{-\infty}^{\infty} g(x) f_{\mathbf{x}}(x) dx. \tag{12}$$

If we are able to normalize the density, then the result will be $q_{\mathcal{D}}$. The complementary distribution function may be written in terms of the integral of this function and we will write

$$\mathrm{Q}_{\mathcal{D}}(y) = \int_{y}^{\infty} q_{\mathcal{D}}(u) du. \tag{13}$$

Table 10 lists the acronyms in one tidy convenient place.

Table 10: The Soup

Acronym	Definition
rv	Random Variable
pdf	Probability density function
cdf	Cumulative distribution function
mgf	Moment generating function
cf	Characteristic function

Finally we define a number of functions.

- The characteristic function of a set S is

$$\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}. \tag{14}$$

- The function $\Gamma(\nu)$ is the gamma function as described in [7] or [8]. Similarly, from the same references, the incomplete gamma function is $\Gamma(\nu, x)$.
- We use the Bessel functions and modified Bessel functions $\mathrm{J}_\nu(z)$ and $\mathrm{I}_\nu(z)$ as defined in [7] or [8].
- The confluent hypergeometric function ${}_1\mathrm{F}_1(a, b; z)$ is found in [7] as $M(a, b, z)$ and in [8] as $\Phi(a, b, z)$.
- L_n^α are the generalized Laguerre polynomials [7].

- The generalized Marcum Q function is

$$\mathrm{Q}_{n/2}(m, x) = \int_x^\infty u^{n/2} e^{-\frac{m^2+u^2}{2}} \frac{I_{n/2-1}(mu)}{m^{n/2-1}} du. \tag{15}$$

  The generalized Marcum Q function is normally defined only for n even, and Marcum's Q function is Q_1. There is no harm in defining it for arbitrary n, so we will do so. However *computation* of Marcum's Q function is much more tractable when n is even.

3 Uniform Random Variables

A random variable $\mathbf{x}$ is uniform, written $\mathbf{x} \sim \mathcal{U}(a,b)$, if it has density

$$f_{\mathcal{U}}(x;a,b) = \frac{1}{b-a}\chi_{[a,b]}(x). \tag{16}$$

The moment generating function is trivially

$$\mathrm{M}_{\mathcal{U}}(s;a,b) = Ee^{s\mathbf{x}} = \frac{e^{sb}-e^{sa}}{(b-a)s}. \tag{17}$$

The moments are found by expanding the mgf in a series

$$\mathrm{M}_{\mathcal{U}}(s;a,b) = \frac{1}{(b-a)s}\sum_{k=0}^{\infty}\left(\frac{(sb)^k-(sa)^k}{k!}\right) = \sum_{k=0}^{\infty}\left(\frac{1}{k+1}\frac{b^{k+1}-a^{k+1}}{b-a}\right)\frac{s^k}{k!} \tag{18}$$

so that

$$E\mathbf{x}^k = \frac{1}{k+1}\frac{b^{k+1}-a^{k+1}}{b-a}. \tag{19}$$

The mean is

$$\mu_{\mathcal{U}} = \frac{b+a}{2}, \tag{20}$$

the second moment

$$E\mathbf{x}^2 = \frac{b^2-a^2}{3(b-a)} = \frac{b^2+ab+a^2}{3}, \tag{21}$$

and variance

$$\sigma^2_{\mathcal{U}} = \frac{(b-a)^2}{12}. \tag{22}$$

A Q function for uniform random variables is

$$\mathrm{Q}_{\mathcal{U}}(x) = \begin{cases} 0 & x < 0 \\ x & x \in [0,1] \\ 1 & x > 1 \end{cases} \tag{23}$$

and the complementary cdf is

$$P\{\mathcal{U}(a,b) > y\} = \mathrm{Q}\left(\frac{y-a}{b-a}\right). \tag{24}$$

Uniform random variables may be used to generate random variables from other distributions. If $\mathbf{x}$ is a r.v. with cdf $F(x)$, then trivially

$$\mathbf{y} = F(\mathbf{x}) \sim \mathcal{U}(0,1). \tag{25}$$

From this if $\mathbf{u} \sim \mathcal{U}(0,1)$,

$$\mathbf{x} = F^{-1}(\mathbf{u}) \tag{26}$$

will have distribution F. Since F is monotonic it will have an inverse, but the inverse must have a closed form for this to be useful.

4 Normal Random Variables

A random variable $\mathbf{x}$ is normal, written $\mathbf{x} \sim \mathcal{N}(\mu, \sigma^2)$, if it has density

$$f_{\mathcal{N}}(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{27}$$

The moment generating function (mgf) of $\mathbf{x}$ is found by completing the square in the exponential of

$$\mathrm{M}_{\mathcal{N}}(s; \mu, \sigma^2) = Ee^{s\mathbf{x}} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{sx} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \tag{28}$$

yielding

$$\mathrm{M}_{\mathcal{N}}(s; \mu, \sigma^2) = e^{\mu s + \frac{\sigma^2 s^2}{2}}. \tag{29}$$

Computing the moments directly or using the mgf is incredibly tedious, but a little change of variables makes it really simple. We will compute central moments, and if you need the general moment just expand

$$E(\mathbf{x} + \mu)^k \tag{30}$$

where $\mathbf{x}$ is a zero-mean normal random variable. The k^{th} central moment is

$$E(\mathbf{x} - \mu)^k = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^k e^{-x^2/2\sigma^2} dx. \tag{31}$$

For k, the integrand is an odd function of x, so the odd central moments are zero. For even k, we have

$$E(\mathbf{x} - \mu)^k = \frac{2}{\sqrt{2\pi\sigma^2}} \int_{0}^{\infty} x^k e^{-x^2/2\sigma^2} dx. \tag{32}$$

Making a change of variables, $t = x^2/2\sigma^2$, we have

$$E(\mathbf{x} - \mu)^k = \frac{2^{k/2}\sigma^k}{\sqrt{\pi}} \int_0^{\infty} t^{k/2-1/2} e^{-t} dt = \frac{2^{k/2}\sigma^k}{\sqrt{\pi}} \Gamma(k/2 + 1/2) = 1 \cdot 3 \cdot \ldots \cdot (k-1)\sigma^k. \tag{33}$$

Normal random variables replicate. If $\mathbf{x}_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $\mathbf{x}_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ are independent[2] then $\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 \sim \mathcal{N}(\alpha_1\mu_1 + \alpha_2\mu_2, \alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2)$. This follows immediately from the characteristic function of $\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2$,

$$\begin{aligned}
\mathrm{M}_{\alpha_1\mathbf{x}_1+\alpha_2\mathbf{x}_2}(s) &= E\left[e^{s(\alpha_1\mathbf{x}_1+\alpha_2\mathbf{x}_2)}\right] \\
&= E\left[E\left[e^{s(\alpha_1\mathbf{x}_1+\alpha_2\mathbf{x}_2)}|\mathbf{x}_2\right]\right] \\
&= E\left[e^{s\alpha_1\mathbf{x}_1} e^{s\alpha_2+s^2\alpha_2^2\sigma_2^2/2}\right] \\
&= e^{s(\alpha_1\mu_1+\alpha_2\mu_2)+s^2(\alpha_1^2\sigma_1^2+\alpha_2^2\sigma_2^2)/2}.
\end{aligned} \tag{34}$$

The Q function for normal random variables is well known

$$\mathrm{Q}_{\mathcal{N}}(x) = \mathrm{Q}(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-u^2/2} du \tag{35}$$

and the complementary cdf is

$$P\{\mathcal{N}(\mu, \sigma^2) > y\} = \mathrm{Q}\left(\frac{y-\mu}{\sigma}\right). \tag{36}$$

[2]This may be generalized to $\mathbf{x}_1$ and $\mathbf{x}_2$ jointly normal

4.1 Craig's Q Function

An alternative form of the Q function was derived in [9]. If we artificially introduce a two-dimensional normal random variable, we may write the Q function as

$$\mathrm{Q}(x) = \frac{1}{2\pi}\int_x^\infty \int_{-\infty}^\infty e^{-\frac{x^2+y^2}{2}}\,dydx. \tag{37}$$

If we convert to polar coordinates, then this becomes

$$\mathrm{Q}(x) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \int_{R_0(\theta)}^\infty re^{-\frac{r^2}{2}}\,drd\theta \tag{38}$$

where $R_0(\theta)$ is the distance from the origin to the line $y = x$ for a given value of θ (see Figure 1). Since

$$R_0(\theta) = \frac{x}{\cos\theta} \tag{39}$$

we have

$$\mathrm{Q}(x) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-\frac{x^2}{2\cos^2\theta}}\,d\theta = \frac{1}{\pi}\int_0^{\pi/2} e^{-\frac{x^2}{2\cos^2\theta}}\,d\theta. \tag{40}$$

A change of variables $\theta \leftarrow \pi/2 - \theta$ yields the alternate result

$$\mathrm{Q}(x) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-\frac{x^2}{2\sin^2\theta}}\,d\theta = \frac{1}{\pi}\int_0^{\pi/2} e^{-\frac{x^2}{2\sin^2\theta}}\,d\theta. \tag{41}$$

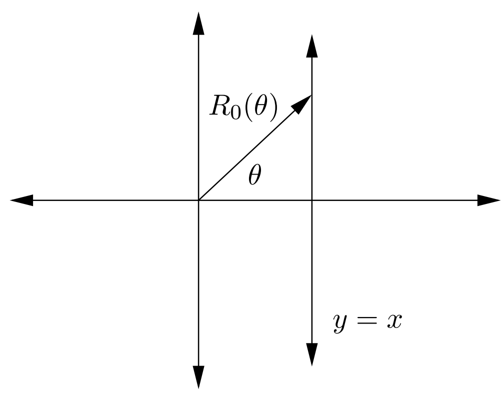

Figure 1: The Q Function in Polar Coordinates

5 Gamma Random Variables

A random variable is a *gamma* random variable if it has the density

$$f_\Gamma(x;\nu,\beta) = \frac{1}{\beta^\nu\Gamma(\nu)} x^{\nu-1} e^{-x/\beta}\chi_{[0,\infty)}(x) \tag{42}$$

where $\beta > 0$ and $\nu \geq 0$. We write $\mathbf{x} \sim \Gamma(\nu,\beta)$.

If $\mathbf{x} \sim \Gamma(\nu,\beta)$, it has the moment generating function

$$\begin{aligned} \mathrm{M}_\Gamma(s;\nu,\beta) &= Ee^{s\mathbf{x}} \\ &= \frac{1}{\beta^\nu\Gamma(\nu)}\int_0^\infty e^{sx}x^{\nu-1}e^{-x/\beta}dx \\ &= \frac{1}{(1-\beta s)^\nu\Gamma(\nu)}\int_0^\infty y^{\nu-1}e^{-y}dy \\ &= \frac{1}{(1-\beta s)^\nu}. \end{aligned} \tag{43}$$

The moments are easily obtained from the mgf. The k^{th} moment is

$$E\mathbf{x}^k = \nu(\nu+1)(\nu+2)\ldots(\nu+k-1)\beta^k = \beta^k\frac{\Gamma(\nu+k)}{\Gamma(\nu)}. \tag{44}$$

In particular, the mean and variance are

$$\mu_\Gamma = \nu\beta, \tag{45}$$

and

$$\sigma_\Gamma^2 = \nu\beta^2. \tag{46}$$

Gamma random variables replicate; if $\mathbf{x}_1 \sim \Gamma(\nu_1,\beta)$ and $\mathbf{x}_2 \sim \Gamma(\nu_2,\beta)$, are independent gamma random variables, then

$$\mathbf{x}_1 + \mathbf{x}_2 \sim \Gamma(\nu_1+\nu_2,\beta). \tag{47}$$

The simplest approach to proving this is to recognize that the mgf of a sum of independent random variables is the product of the mgfs, so that

$$\mathrm{M}_{\mathbf{x}_1}(s;\nu_1,\beta)\mathrm{M}_{\mathbf{x}_2}(s;\nu_2,\beta) = \frac{1}{(1-\beta s)^{\nu_1+\nu_2}} \tag{48}$$

which inverts to the desired result. This result may be obtained directly by evaluating the convolution integral.

A Q function for gamma random variables is obtained by letting $u = x/\beta$ in the complementary cdf yielding

$$\mathrm{Q}_\Gamma(x;\nu) = \frac{1}{\Gamma(\nu)}\int_x^\infty u^{\nu-1}e^{-u}du = \Gamma(\nu,x). \tag{49}$$

The complementary cdf is

$$P\{\Gamma(\nu,\beta) > y\} = \mathrm{Q}_\Gamma\left(y/\beta;\nu\right). \tag{50}$$

Note that for $y < 0$, $P\{\Gamma(\nu,\beta) > y\} = 1$. For this and all the other distributions that follow we will simply make this implicit.

5.1 Exponential Random Variables

A gamma r.v. $\mathbf{x}$ with $\nu = 1$ is an *exponential* random variable, written $\mathbf{x} \sim \mathcal{E}$ and has density

$$f_{\mathcal{E}}(x; \beta) = \frac{1}{\beta} e^{-x/\beta} \chi_{[0,\infty)}(x). \tag{51}$$

All of the properties of exponential r.v.s follow immediately from the properties of gamma r.v.s.

The Q function is delightfully simple and hardly worth writing down

$$\mathrm{Q}_{\mathcal{E}}(x) = \int_x^\infty e^{-u} du = e^{-x} \tag{52}$$

and the complementary cdf is

$$P\{\mathcal{E}(\beta) > y\} = \mathrm{Q}_{\mathcal{E}}(y/\beta) = e^{-y/\beta}. \tag{53}$$

5.2 Chi-squared Random Variables

If $\mathbf{x} \sim \mathcal{N}(0, \sigma^2)$ and $\mathbf{y} = \mathbf{x}^2$, the density of $\mathbf{y}$ is

$$f_{\mathbf{y}}(y) = \frac{1}{\sqrt{2\pi\sigma^2 y}} e^{-y/2\sigma^2} \chi_{[0,\infty)}(y). \tag{54}$$

This is a gamma r.v. with $\nu = 1/2$ and $\beta = 2\sigma^2$. If, now,

$$\mathbf{y} = \sum_{k=1}^{n} \mathbf{x}_k^2 \tag{55}$$

where the $\mathbf{x}_k$ are i.i.d $\mathcal{N}(0, \sigma^2)$ then $\mathbf{y}$ is a chi-squared (χ^2) random variable with n degrees of freedom written $\mathbf{y} \sim \chi^2(n, \sigma^2)$. We use the replicating property of gamma r.v.s to obtain the $\Gamma(n/2, 2\sigma^2)$ density

$$f_{\chi^2}(y; n, \sigma^2) = \frac{1}{(2\sigma^2)^{n/2}\Gamma(n/2)} y^{n/2-1} e^{-y/2\sigma^2} \chi_{[0,\infty)}(y) \tag{56}$$

and mgf

$$\mathrm{M}_{\chi^2}(s; n, \sigma^2) = \frac{1}{(1 - 2\sigma^2 s)^{n/2}}. \tag{57}$$

When n is even, the cumulative distribution function of a chi-square random variable may be computed by repeated integration by parts:

$$F_{\chi^2}(y) = \int_0^y \frac{1}{(2\sigma^2)^{n/2}\Gamma(n/2)} u^{n/2-1} e^{-u/2\sigma^2} du = 1 - e^{-y/2\sigma^2} \sum_{k=0}^{n/2-1} \frac{1}{k!} \left(\frac{y}{2\sigma^2}\right)^k. \tag{58}$$

A Q function for a χ^2 random variables is

$$\mathrm{Q}_{\chi^2}(x; n) = \frac{1}{\Gamma(n/2)} \int_x^\infty u^{n/2-1} e^{-u} du = \Gamma(n/2, x). \tag{59}$$

The complementary cdf is

$$P\{\chi^2(n, \sigma^2) > y\} = \mathrm{Q}_{\chi^2}\left(\frac{y}{2\sigma^2}; n\right) = \Gamma(n/2, y/2\sigma^2). \tag{60}$$

For n even,

$$\mathrm{Q}_{\chi^2}(x; n) = e^{-x} \sum_{k=0}^{n/2-1} \frac{x^k}{k!}. \tag{61}$$

6 Noncentral χ^2 Random Variables

If $\mathbf{x} \sim \mathcal{N}(\mu, \sigma^2)$ and $\mathbf{y} = \mathbf{x}^2$, the density of $\mathbf{y}$ is found using the same method as for χ^2 random variables. If $x_1 = +\sqrt{y}$ and $x_2 = -\sqrt{y}$, then

$$f_{\mathbf{y}}(y) = \frac{f_{\mathbf{x}}(x_1)}{2|x_1|} + \frac{f_{\mathbf{x}}(x_2)}{2|x_2|} \tag{62}$$

so that

$$\begin{aligned} f_{\mathbf{y}}(y) &= \frac{1}{\sqrt{2\pi\sigma^2}} \frac{e^{(\sqrt{y}-\mu)^2/2\sigma^2} + e^{(-\sqrt{y}-\mu)^2/2\sigma^2}}{2\sqrt{y}} \\ &= \frac{1}{\sqrt{2\pi\sigma^2 y}} e^{-(y+\mu^2)/2\sigma^2} \cosh\left(\frac{\sqrt{y\mu^2}}{\sigma^2}\right) \chi_{[0,\infty)}(y). \end{aligned} \tag{63}$$

Since

$$\cosh(x) = \sqrt{\frac{\pi x}{2}} \mathrm{I}_{-1/2}(x) \tag{64}$$

we may write the density as

$$f_{\mathbf{y}}(y) = \frac{1}{2\sigma^2} \left(\frac{y}{\mu^2}\right)^{-1/4} e^{-(\mu^2+y)/2\sigma^2} \mathrm{I}_{-1/2}\left(\frac{\sqrt{y\mu^2}}{\sigma^2}\right) \chi_{(0,\infty]}(y). \tag{65}$$

The mgf of y is

$$\mathrm{M}_{\mathbf{y}}(s) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^\infty e^{sx^2} e^{-(x-\mu)^2/2\sigma^2} dx. \tag{66}$$

Combining the exponential terms and completing the square, we obtain

$$\mathrm{M}_{\mathbf{y}}(s) = \frac{1}{\sqrt{1-2s\sigma^2}} e^{\frac{\mu^2 s}{1-2s\sigma^2}}. \tag{67}$$

If $\mathbf{x}_k \sim \mathcal{N}(\mu_k, \sigma^2)$ are independent and

$$\mathbf{y} = \sum_{k=1}^{n} \mathbf{x}_k^2 \tag{68}$$

then $\mathbf{y}$ is a *noncentral chi-squared* r.v., written $\mathbf{y} \sim \bar{\chi}^2(n, \mu^2, \sigma^2)$ and has mgf

$$\mathrm{M}_{\bar{\chi}^2}(s; n, \mu^2, \sigma^2) = \frac{1}{(1-2s\sigma^2)^{n/2}} e^{\frac{\mu^2 s}{1-2s\sigma^2}} \tag{69}$$

where

$$\mu^2 = \sum_{k=0}^{n} \mu_k^2. \tag{70}$$

From [7] we have the Laplace transform

$$\frac{1}{s^\mu} e^{\frac{k}{s}} \rightleftharpoons \left(\frac{t}{k}\right)^{\frac{\mu-1}{2}} \mathrm{I}_{\mu-1}\left(2\sqrt{kt}\right) \tag{71}$$

If we use elementary Laplace transform operations to manipulate the mgf, we find the density of **y** to be

$$f_{\bar{\chi}^2}(y; n, \mu^2, \sigma^2) = \frac{1}{2\sigma^2}\left(\frac{y}{\mu^2}\right)^{(n-2)/4} e^{-(\mu^2+y)/2\sigma^2} \mathrm{I}_{n/2-1}\left(\frac{\sqrt{y\mu^2}}{\sigma^2}\right)\chi_{[0,\infty)}(y). \tag{72}$$

The mean and variance may be found by differentiating the mgf, resulting in

$$\mu_{\bar{\chi}^2} = n\sigma^2 + \mu^2, \tag{73}$$

and

$$\sigma^2_{\bar{\chi}^2} = 2n\sigma^4 + 4\sigma^2\mu^2. \tag{74}$$

In general, the moments of the noncentral χ^2 random variable may be found in terms of confluent hypergeometric functions. We base our derivation on the integral (see [8], paragraph 9.211)

$${}_1\mathrm{F}_1(-\nu, \alpha+1; z) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\nu+1)} e^z z^{-\alpha/2} \int_0^\infty e^{-t} t^{\nu+\alpha/2} J_\alpha(2\sqrt{zt})dt \tag{75}$$

where ${}_1\mathrm{F}_1(\alpha, \beta; x)$ is a confluent hypergeometric function[7],[8]. The k^{th} moment is

$$E\mathbf{x}^k = \frac{1}{2\sigma^2}\int_0^\infty y^k \left(\frac{y}{\mu^2}\right)^{(n-2)/4} e^{-(\mu^2+y)/2\sigma^2} \mathrm{I}_{n/2-1}\left(\frac{\sqrt{y\mu^2}}{\sigma^2}\right) dy. \tag{76}$$

If we let $\alpha = n/2 - 1$, $t = y/2\sigma^2$, $m^2 = \mu^2/2\sigma^2$, and $z = -m^2$ we can write the moment as

$$E\mathbf{x}^k = \frac{(2\sigma^2)^k e^z i^{-\alpha}}{m^\alpha}\int_0^\infty t^{k+\alpha/2} e^{-t} \mathrm{J}_\alpha(2\sqrt{zt})dt. \tag{77}$$

Applying the above integral and fiddling a bit we obtain

$$E\mathbf{x}^k = (2\sigma^2)^k \frac{\Gamma(k+n/2)}{\Gamma(n/2)} {}_1\mathrm{F}_1(-k, n/2; -\mu^2/2\sigma^2). \tag{78}$$

If n is an even integer this reduces to [7]

$$E\mathbf{x}^k = (2\sigma^2)^k k! L_k^{(n/2-1)}(-\mu^2/2\sigma^2). \tag{79}$$

A confluent hypergeometric function ${}_1\mathrm{F}_1(-k, \beta; x)$ with k a positive integer is a polynomial in x of degree k, so the moments are polynomials in σ^2. In particular, the first moment is

$$\begin{aligned} E\mathbf{x} &= (2\sigma^2)\frac{\Gamma(1+n/2)}{\Gamma(n/2)}\left[1 + \frac{-1}{n/2}\frac{-\mu^2}{2\sigma^2}\right] \\ &= n\sigma^2 + \mu^2. \end{aligned} \tag{80}$$

The second moment is

$$\begin{aligned} E\mathbf{x}^2 &= (4\sigma^4)\frac{\Gamma(2+n/2)}{\Gamma(n/2)}\left[1 + \frac{-2}{n/2}\frac{-\mu^2}{2\sigma^2} + \frac{(-2)(-1)}{(n/2)(n/2+1)}\frac{\mu^4}{2!4\sigma^4}\right] \\ &= n(n+2)\sigma^4 + 2(n+2)\mu^2\sigma^2 + \mu^4. \end{aligned} \tag{81}$$

When n is an even integer, an alternate approach yields the same results. If we let $M = n/2$ and $t = 2s\sigma^2$ then we may write the normalized mgf as

$$\mathrm{M}_{\bar{\chi}^2}(t/2\sigma^2) = \frac{1}{(1-t)^M} e^{\frac{-(-m^2)t}{1-t}}. \tag{82}$$

If M is an integer, then this last term is the generating function for the associated Laguerre polynomials, so that

$$\begin{aligned} \mathrm{M}_{\bar{\chi}^2}(t/2\sigma^2) &= \sum_{k=0}^{\infty} L_k^{(M-1)}(-m^2)t^k \\ &= \sum_{k=0}^{\infty} k! L_k^{(M-1)}(-m^2)\frac{t^k}{k!}. \end{aligned} \tag{83}$$

Reversing the normalization we have the n^{th} moment

$$E\mathbf{y}^k = (2\sigma^2)^k k! L_k^{(M-1)}(-\mu^2/2\sigma^2). \tag{84}$$

A Q function may be found for noncentral χ^2 random variables by letting $w = u/2\sigma^2$ and $m^2 = \mu^2/2\sigma^2$ in the complementary cdf. Note that we could normalize by σ^2, but normalizing by $1/2\sigma^2$ agrees with the normalization for the central χ^2 Q function. With this

$$P\{\bar{\chi}^2(n,\mu^2,\sigma^2) > y\} = \int_{y/2\sigma^2}^{\infty} (\sqrt{w})^{n/2-1} e^{-(m^2+w)} \frac{\mathrm{I}_{n/2-1}\left(2\sqrt{m^2 w}\right)}{m^{n/2-1}} dw. \tag{85}$$

Thus, we may define the Q function to be

$$\mathrm{Q}_{\bar{\chi}^2}(x;n,m^2) = \int_{x}^{\infty} (\sqrt{w})^{n/2-1} e^{-(m^2+w)} \frac{\mathrm{I}_{n/2-1}\left(2\sqrt{m^2 w}\right)}{m^{n/2-1}} dw. \tag{86}$$

Of course

$$P\{\bar{\chi}^2(n,\mu^2,\sigma^2) > y\} = \mathrm{Q}_{\bar{\chi}^2}(y/2\sigma^2; n, \mu^2/2\sigma^2). \tag{87}$$

7 Rayleigh Random Variables

If $\mathbf{x}_k$ are i.i.d $\mathcal{N}(0, \sigma^2)$ and

$$\mathbf{y} = \sqrt{\sum_{k=1}^{n} x_k^2} \tag{88}$$

then $\mathbf{y}$ is a Rayleigh random variable, written $\mathbf{y} \sim \mathcal{R}(n, \sigma^2)$. Since

$$\sum_{k=1}^{n} x_k^2 \sim \chi^2(n, \sigma^2) \tag{89}$$

the density of $\mathbf{y}$ is

$$f_{\mathcal{R}}(y; n, \sigma^2) = 2y f_{\chi^2}(y^2) = \frac{2}{(2\sigma^2)^{n/2}\Gamma(n/2)} y^{n-1} e^{-y^2/2\sigma^2} \chi_{[0,\infty)}. \tag{90}$$

The k^{th} moment of a Rayleigh r.v. $\mathbf{x}$ is

$$E\mathbf{x}^k = \frac{2}{(2\sigma^2)^{n/2}\Gamma(n/2)} \int_0^\infty u^k u^{n-1} e^{-u^2/2\sigma^2} du. \tag{91}$$

Setting $t = u^2/\sqrt{2\sigma^2}$

$$E\mathbf{x}^k = \frac{(2\sigma^2)^{k/2}}{\Gamma(n/2)} \int_0^\infty t^{\frac{n+k}{2}-1} e^{-t} dt = (2\sigma^2)^{k/2} \frac{\Gamma((n+k)/2)}{\Gamma(n/2)}. \tag{92}$$

A Q function is found by setting $u = y/\sigma$ yielding

$$P\{\mathcal{R}(n, \sigma^2) > y\} = \frac{1}{2^{n/2-1}\Gamma(n/2)} \int_{y/\sigma}^\infty u^{n-1} e^{-u^2/2} du \tag{93}$$

so

$$\mathrm{Q}_{\mathcal{R}}(x; n) = \frac{1}{2^{n/2-1}\Gamma(n/2)} \int_x^\infty u^{n-1} e^{-u^2/2} du \tag{94}$$

and

$$P\{\mathcal{R}(n, \sigma^2) > y\} = \mathrm{Q}_{\mathcal{R}}\left(\frac{y}{\sigma}\right). \tag{95}$$

Note that the Q function may be written in the form of the incomplete gamma function by making the subsitution $t = u^2/2$ yielding

$$\mathrm{Q}_{\mathcal{R}}(x; n) = \frac{1}{\Gamma(n/2)} \int_{x^2/2}^\infty t^{n/2-1} e^{-t} dt = \Gamma(n/2, x^2/2). \tag{96}$$

For n even,

$$\mathrm{Q}_{\mathcal{R}}(x; n) = e^{-x^2/2} \sum_{k=0}^{n/2-1} \frac{x^{2k}}{k!}. \tag{97}$$

8 Rice Random Variables

If $\mathbf{x}_k$ are independent $\mathcal{N}(\mu_k, \sigma^2)$ and

$$\mathbf{y} = \sqrt{\sum_{k=1}^{n} x_k^2} \tag{98}$$

then $\mathbf{y}$ has a Rice distribution (also called a Rice-Nakagami distribution), written $\mathbf{y} \sim \bar{\mathcal{R}}(n, \mu^2, \sigma^2)$ where

$$\mu^2 = \sum_{k=0}^{n} \mu_k^2. \tag{99}$$

Since

$$\sum_{k=1}^{n} x_k^2 \sim \bar{\chi}^2(n, \mu^2\sigma^2) \tag{100}$$

the density of $\mathbf{y}$ is found to be

$$f_{\bar{\mathcal{R}}}(y; n, \mu^2, \sigma^2) = 2y f_{\bar{\chi}^2}(y^2) = \frac{1}{\sigma^2} y^{n/2} e^{-\frac{\mu^2+y^2}{2\sigma^2}} \frac{\mathrm{I}_{n/2-1}\left(\frac{\mu y}{\sigma^2}\right)}{\mu^{n/2-1}} \chi_{(0,\infty]}(y). \tag{101}$$

The moments are found in terms of confluent hypergeometric functions, based on the integral ([8], paragraph 6.631)

$$\int_0^\infty x^\mu e^{-\alpha x^2} \mathrm{J}_\nu(\beta x) dx = \frac{\beta^\nu}{2^{\nu+1}\alpha^{(\mu+\nu+1)/2}} \frac{\Gamma((\mu+\nu+1)/2)}{\Gamma(\nu+1)} {}_1\mathrm{F}_1\left(\frac{\mu+\nu+1}{2}, \nu+1; -\frac{\beta^2}{4\alpha}\right). \tag{102}$$

The k^{th} moment is

$$E\mathbf{x}^k = \frac{1}{\sigma^2} \int_0^\infty y^{k+n/2} e^{-\frac{\mu^2+y^2}{2\sigma^2}} \frac{\mathrm{I}_{n/2-1}\left(\frac{\mu y}{\sigma^2}\right)}{\mu^{n/2-1}} dy. \tag{103}$$

If we let $t = y/\sigma$, $m = \mu/\sigma$, and $\beta = im$ then

$$E\mathbf{x}^k = \frac{\sigma^k e^{-m^2/2}}{i^{n/2-1} m^{n/2-1}} \int_0^\infty t^{k+n/2} e^{-t^2/2} \mathrm{J}_{n/2-1}(\beta t) dt. \tag{104}$$

Applying the integral and fiddling a bit we obtain

$$E\mathbf{x}^k = (2\sigma^2)^{k/2} e^{-\mu^2/2\sigma^2} \frac{\Gamma((n+k)/2)}{\Gamma(n/2)} {}_1\mathrm{F}_1\left(\frac{n+k}{2}, \frac{n}{2}; \frac{\mu^2}{2\sigma^2}\right). \tag{105}$$

A Q function is found by setting $u = y/\sigma$ and $m = \mu/\sigma$ yielding

$$P\{\bar{\mathcal{R}}(n, \mu^2, \sigma^2 > y)\} = \int_{y/\sigma}^\infty u^{n/2} e^{-\frac{m^2+u^2}{2}} \frac{I_{n/2-1}(mu)}{m^{n/2-1}} du. \tag{106}$$

With this the tail probability is found in terms of Marcum's Q function

$$\mathrm{Q}_{\bar{\mathcal{R}}}(x; n, m^2) = \mathrm{Q}_{n/2}(m, x) = \int_x^\infty u^{n/2} e^{-\frac{m^2+u^2}{2}} \frac{I_{n/2-1}(mu)}{m^{n/2-1}} du. \tag{107}$$

and

$$P\{\bar{\mathcal{R}}(n, \mu^2, \sigma^2) > y\} = \mathrm{Q}_{n/2}\left(\frac{m}{\sigma}, \frac{y}{\sigma}\right). \tag{108}$$

9 Connections Between χ^2, Rayleigh, Noncentral χ^2 and Rice Random Variables

The Q function for noncentral χ^2 and Marcum's Q function may each be expressed in terms of the other by making simple variable substitutions. Thus

$$\mathrm{Q}_{\bar{\chi}^2}(x; n, m^2) = \mathrm{Q}_{n/2}\left(\sqrt{2m^2}, \sqrt{2x}\right) \tag{109}$$

and

$$\mathrm{Q}_{n/2}(\alpha, \beta) = \mathrm{Q}_{\bar{\chi}^2}\left(\beta^2/2; n, \alpha^2/2\right). \tag{110}$$

This allows us to express tail probabilities

$$P\{\bar{\chi}^2(n, \mu^2, \sigma^2) > y\} = \mathrm{Q}_{\bar{\chi}^2}\left(y/2\sigma^2; n, \mu^2/2\sigma^2\right) = \mathrm{Q}_{n/2}\left(\sqrt{\mu^2/\sigma^2}, \sqrt{y/\sigma^2}\right) \tag{111}$$

and

$$P\{\bar{\mathcal{R}}(n, \mu^2, \sigma^2) > y\} = \mathrm{Q}_{n/2}\left(\mu/\sigma, y/\sigma\right) = \mathrm{Q}_{\bar{\chi}^2}\left(y^2/2\sigma^2; n, \mu^2/2\sigma^2\right). \tag{112}$$

In a completely analogous fashion we may connect the Q functions of the χ^2 and Rayleigh random variables.

$$P\{\chi^2(n, \sigma^2) > y\} = \mathrm{Q}_{\chi^2}\left(y/2\sigma^2; n\right) = \mathrm{Q}_{\mathcal{R}}\left(\sqrt{y/\sigma^2}; n\right) \tag{113}$$

and

$$P\{\mathcal{R}(n, \sigma^2) > y\} = \mathrm{Q}_{\mathcal{R}}\left(y/\sigma; n\right) = \mathrm{Q}_{\chi^2}\left(y^2/2\sigma^2; n\right). \tag{114}$$

It is also clear from the definitions that the χ^2 rv is a special case of the noncentral χ^2 rv when the noncentrality parameter is $\mu^2 = 0$. This is also clear by inspecting the mgf for the noncentral χ^2 random variable. The density is less clear, but if we recognize that

$$\lim_{\alpha \to 0} \frac{\mathrm{I}_\nu(\alpha z)}{\alpha^\nu} = \frac{z^\nu}{2^\nu \Gamma(\nu + 1)}. \tag{115}$$

then the result is immediate. Similarly, a Rayleigh rv is a special case of the Rice rv and the same process allows us to special-case the density function.

10 Lognormal Random Variables

If $\mathbf{x} \sim \mathcal{N}(\mu, \sigma^2)$ and

$$\mathbf{y} = \alpha^{\mathbf{x}/\beta} \tag{116}$$

with $\alpha > 1$ and $\beta \geq 1$ then $\mathbf{y}$ is said to be a *lognormal* random variable, $\mathcal{L}(\mu, \sigma, \alpha, \beta)$. In this section we explore the properties of the lognormal random variable.

The density of $\mathbf{y}$ is found by writing[2]

$$y = g(x) = \alpha^{x/\beta}, \tag{117}$$

so that

$$x = \beta \log_\alpha y \quad \text{and} \quad g' = \frac{\beta \log \alpha}{y} \tag{118}$$

and finally

$$f_{\mathcal{L}}(y) = \frac{f_{\mathbf{x}}(x)}{g'(x)}. \tag{119}$$

Combining this mess,

$$f_{\mathcal{L}}(y; \mu, \sigma^2, \alpha, \beta) = \frac{1}{\gamma} \frac{1}{y\sqrt{2\pi\sigma^2}} e^{-(\beta \log_\alpha y - \mu)^2/2\sigma^2} \chi_{[0,\infty)}(y) \tag{120}$$

where

$$\gamma = \log \alpha^{\frac{1}{\beta}} = \frac{1}{\beta} \log \alpha. \tag{121}$$

The k^{th} moment of a lognormal R.V. is found by evaluating

$$E\mathbf{y}^k = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \alpha^{kx/\beta} e^{-(x-\mu)^2/2\sigma^2} dx \tag{122}$$

writing

$$\alpha^{kx/\beta} = e^{kx\gamma} \tag{123}$$

then combining the terms and completing the square in the exponential yields

$$E\mathbf{y}^k = \alpha^{k\mu/\beta} e^{\frac{1}{2}\left[\log \alpha^{k\sigma/\beta}\right]^2} \tag{124}$$

If we define

$$\tilde{\mu} = \alpha^{\mu/\beta} \tag{125}$$

and

$$\tilde{\sigma} = \alpha^{\sigma/\beta} \tag{126}$$

then

$$E\mathbf{y}^k = \tilde{\mu}^k e^{\frac{1}{2}\left[\log \tilde{\sigma}^k\right]^2}. \tag{127}$$

The lognormal distribution is used in some specific ways. Normally, mathematicians will use

$$\mathbf{y} = e^{\mathbf{x}} \tag{128}$$

so that the density is

$$f_{\mathcal{L}}(y; \mu, \sigma^2, e, 1) = \frac{1}{y\sqrt{2\pi\sigma^2}} e^{-(\log y - \mu)^2/2\sigma^2} \chi_{[0,\infty)}(y). \tag{129}$$

In this case, $\tilde{\mu} = \mu$ and $\tilde{\sigma} = \sigma$. Engineers, on the other hand, will frequently use

$$\mathbf{y} = 10^{\mathbf{x}/20} \tag{130}$$

as the definition for a lognormal distribution. In this case

$$f_{\mathcal{L}}(y; \mu, \sigma^2, 10, 20) = \frac{20}{\log 10} \frac{1}{y\sqrt{2\pi\sigma^2}} e^{-(20\log_{10} y - \mu)^2/2\sigma^2} \chi_{[0,\infty)}(y). \tag{131}$$

In this case $\tilde{\mu} = 10^{\mu/20}$ and $\tilde{\sigma} = 10^{\sigma/20}$. These parameters are normally specified by providing μ and σ in dB.

References

[1] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*. New York: McGraw-Hill, 1975.

[2] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*. New York: McGraw-Hill, third edition ed., 1991.

[3] V. K. Rohatgi, *An Introduction to Probability and Mathematical Statistics*. New York: John Wiley and Sons, 1976.

[4] A. M. Mood, F. A. Graybill, and D. C. Boes, *Introduction to the Theory of Statistics*. New York: McGraw-Hill, second ed., 1974.

[5] W. Feller, *An Introduction to Probability Theory and its Applications, Volume I*. New York: Wiley, 1966.

[6] J. G. Proakis, *Digital Communications*. New York: McGraw-Hill, third ed., 1995.

[7] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. New York: Dover Publications, Inc., 1972.

[8] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*. Academic Press, 1980.

[9] J. W. Craig, "A new, simple and exact result for calculating the probability of error for two-dimensional signal constellations," *Proceedings of the 1991 IEEE Conference on Military Communications*, 1991.