PhaseLocked Systems

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Computation of Rice and Noncentral Chi-Squared Probabilities

Robert T. Short
PhaseLocked Systems
rtshort@ieee.org

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Abstract

This report is a discussion of methods for computing the generalized Marcum Q function. Included is a summary of noncentral χ^2 and Rice random variables, some properties of Marcum's Q function, a discussion of inverting the Q function and details of the methods themselves. A Bessel function series approach, Parl's method, and Dillard's method are all included along with performance comparisons.

Forward

Recently, as part of an analysis task, I found for the umpteenth time in my career that I needed a routine to compute Marcum's Q function and for the umpteenth time I didn't have one. One would think that computing probabilities for Rice and noncentral χ^2 random variables would be old hat by now. After all Rice and Marcum published their work (e.g. [1]) by 1950. I went off to find a good algorithm, preferably already coded, and ended up spending considerable time digging into the problem (and having some fun while I was at it). The result is a set of `octave`[2] routines that compute the Marcum Q function.

While I did accomplish what I set out to, I have implemented only three of the bejillions of algorithms that are published in the literature and I am not entirely happy with the results. I spent more time on this than I really had available, but it is interesting stuff and my intent is to spend time (carefully rationed) in the future implementing other algorithms and possibly correcting the deficiencies in the algorithms I find.

This report is really just a "lab notebook" entry to remind myself of what I have uncovered. I don't believe there is anything new in here since all I am doing is perusing the literature to find good algorithms. I am making this report public in case it may be useful to someone else, and just perhaps comments and corrections will prove useful to me.

This report, a related report[3] and some `octave` samples are available from my website `http://www.phaselockedsystems.com/publications`. Comments and corrections are welcome and may be sent to the email address on the cover page.

1 Introduction and Summary

This report is a discussion of methods for computing the generalized Marcum Q function. Derivations of the methods and performance comparisons are included.

This paper is broken into several sections. In addition to this short introduction, there is a section that very briefly reviews noncentral χ^2 and Rice random variables, followed by a section describing the Marcum Q function. Then sections detailing algorithms for computing the Q function including

- A series of Bessel functions,
- Parl's method[4] and
- Dillard's method[5].

The impatient reader may just want to skip to Section 6 where some performance comparisons are made.

2 Noncentral χ^2 and Rice Random Variables

Noncentral χ^2 and Rice random variables are closely connected, and the probabilities for both may be computed using the same algorithms. This section very briefly reviews both.

2.1 Noncentral χ^2 Random Variables

Recall that if $\mathbf{x}_k \sim \mathcal{N}(\mu_k, \sigma^2)$, for $k = 1 \ldots n$ and n a positive integer are independent and

$$\mathbf{y} = \sum_{k=1}^{n} \mathbf{x}_k^2 \tag{1}$$

then $\mathbf{y}$ is a noncentral chi-squared r.v. and has mgf

$$\mathrm{M}_{\bar{\chi}^2}(s; n, \mu^2, \sigma^2) = \frac{1}{(1-2s\sigma^2)^{n/2}} e^{\frac{\mu^2 s}{1-2s\sigma^2}}. \tag{2}$$

where

$$\mu^2 = \sum_{k=0}^{n} \mu_k^2. \tag{3}$$

The density of the noncentral χ^2 random variable is

$$f_{\bar{\chi}^2}(y; n, \mu^2, \sigma^2) = \frac{1}{2\sigma^2} \left(\frac{y}{\mu^2}\right)^{(n-2)/4} e^{-(\mu^2+y)/2\sigma^2} \mathrm{I}_{n/2-1}\left(\frac{\sqrt{y\mu^2}}{\sigma^2}\right) \chi_{(0,\infty]}(y). \tag{4}$$

The mean and variance are

$$\mu_{\bar{\chi}^2} = n\sigma^2 + \mu^2, \tag{5}$$

and

$$\sigma^2_{\bar{\chi}^2} = 2n\sigma^2 + 4\sigma^2\mu^2. \tag{6}$$

A Q function for noncentral χ^2 random variables is

$$\mathrm{Q}_{\bar{\chi}^2}(x; n, m^2) = \int_x^\infty (\sqrt{w})^{n/2-1} e^{-(m^2+w)} \frac{\mathrm{I}_{n/2-1}\left(2\sqrt{m^2 w}\right)}{m^{n/2-1}} dw \tag{7}$$

and tail probabilities may be found from the Q function

$$P\{\bar{\chi}^2(n, \mu, \sigma^2) > y\} = \mathrm{Q}_{\bar{\chi}^2}(y/2\sigma^2; n, \mu^2/2\sigma^2). \tag{8}$$

The normalized density function (the kernel of the Q function) is plotted in Figures 1, 2, 3, and 3 for various values of the parameters . The vertical hashmark on each trace is the mean of the distribution.

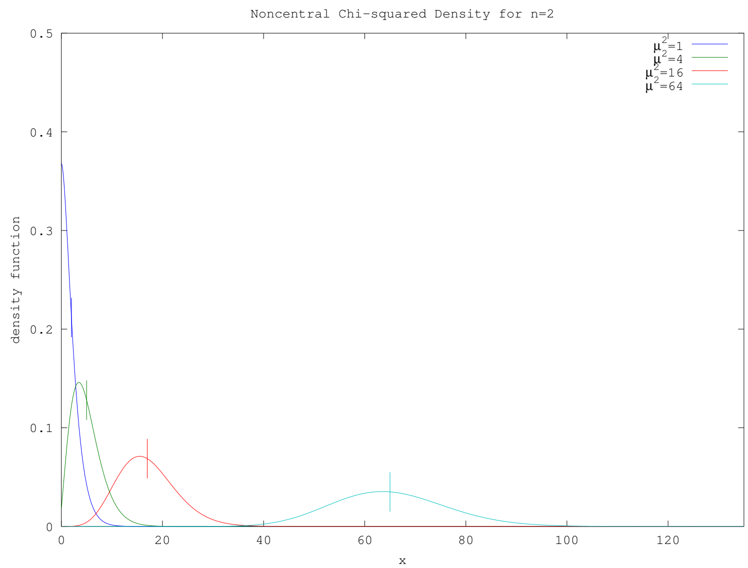

Figure 1: Noncentral χ^2 Density for $n = 2$

2.2 Rice Random Variables

If $\mathbf{x}_k$ are independent $\mathcal{N}(\mu_k, \sigma^2)$ with $k = 1 \ldots n$ and n a positive integer and

$$\mathbf{y} = \sqrt{\sum_{k=1}^{n} x_k^2} \tag{9}$$

then $\mathbf{y}$ has a Rice distribution.

The density of $\mathbf{y}$ is

$$f_{\bar{\mathcal{R}}}(y; n, \mu^2, \sigma^2) = \frac{1}{\sigma^2} y^{n/2} e^{-\frac{\mu^2+y^2}{2\sigma^2}} \frac{\mathrm{I}_{n/2-1}\left(\frac{\mu y}{\sigma^2}\right)}{\mu^{n/2-1}} \chi_{(0,\infty]}(y) \tag{10}$$

where

$$\mu^2 = \sum_{k=0}^{n} \mu_k^2. \tag{11}$$

The k^{th} moment is

$$E\mathbf{x}^k = (2\sigma^2)^{k/2} e^{-\mu^2/2\sigma^2} \frac{\Gamma((n+k)/2)}{\Gamma(n/2)} \; {}_1F_1\left(\frac{n+k}{2}, \frac{n}{2}; \frac{\mu^2}{2\sigma^2}\right). \tag{12}$$

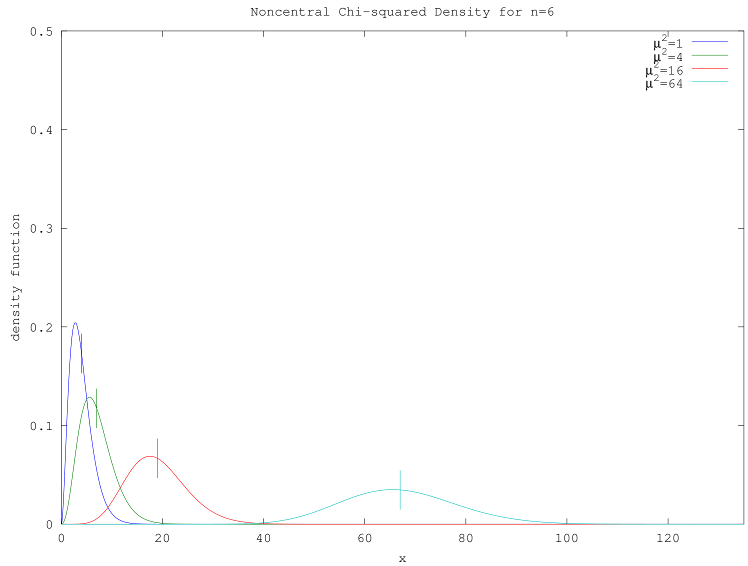

Figure 2: Noncentral χ^2 Density for $n = 6$

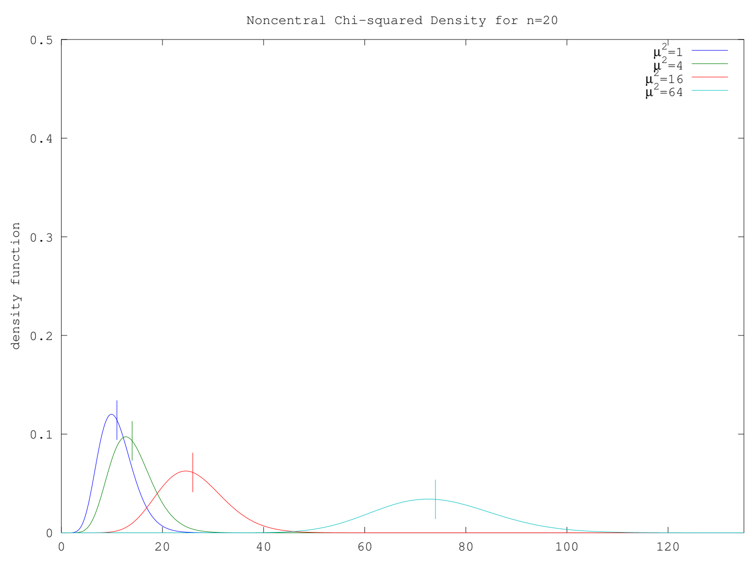

Figure 3: Noncentral χ^2 Density for $n = 20$

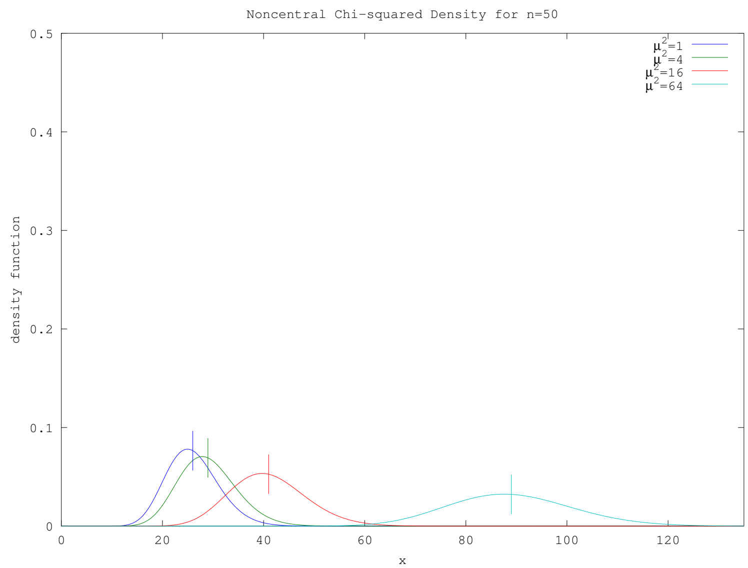

Figure 4: Noncentral χ^2 Density for $n = 50$

The tail probabilities are found in terms of Marcum's Q function[1]

$$\mathrm{Q}_{\bar{\mathcal{R}}}(x; n, m) = \mathrm{Q}_{n/2}(m, x) = \int_x^\infty u^{n/2} e^{-\frac{m^2+u^2}{2}} \frac{I_{n/2-1}(mu)}{m^{n/2-1}} du. \tag{13}$$

and

$$P\{\bar{\mathcal{R}}(n, \mu, \sigma^2) > y\} = \mathrm{Q}_{n/2}\left(\frac{\mu}{\sigma}, \frac{y}{\sigma}\right). \tag{14}$$

The normalized Rice density (the kernel of Marcum's Q function) is plotted in Figures 5, 6, 7, and 8. Again the vertical hashmarks denote the distribution mean.

3 Marcum's Q Function

In Section 2 we presented a Q function for both the noncentral χ^2 distribution and the Rice distribution. As we will see, we may use either Q function to compute the tail probablities for both distributions. In the detection literature it has become fairly common to express results in terms of Marcum's Q function, and in this section we explore Marcum's Q function in some depth.

Marcum's Q function is defined to be [1]

$$\mathrm{Q}(\alpha, \beta) = \int_\beta^\infty \nu e^{-\frac{\nu^2+\alpha^2}{2}} \mathrm{I}_0(\alpha\nu) d\nu \tag{15}$$

and the generalized Marcum Q function is

$$\mathrm{Q}_M(\alpha, \beta) = \frac{1}{\alpha^{M-1}} \int_\beta^\infty \nu^M e^{-\frac{\nu^2+\alpha^2}{2}} \mathrm{I}_{M-1}(\alpha\nu) d\nu. \tag{16}$$

Obviously $\mathrm{Q}(\alpha, \beta) = \mathrm{Q}_1(\alpha, \beta)$. We have already seen this function in Equation 13.

[1]Marcum's Q function is normally defined for n even, or $M = n/2$ an integer. There is no harm in *defining* this Q function for any integer n, but all of the computational algorithms we discuss will assume n even.

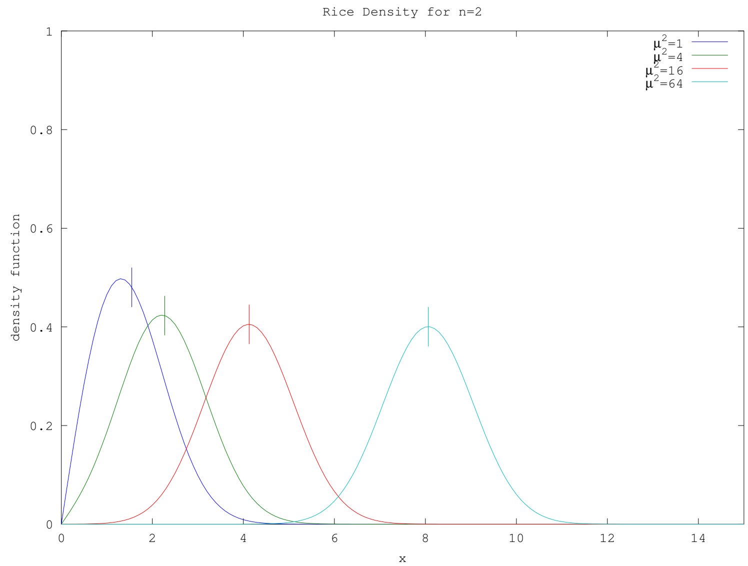

Figure 5: Rice Density for $n = 2$

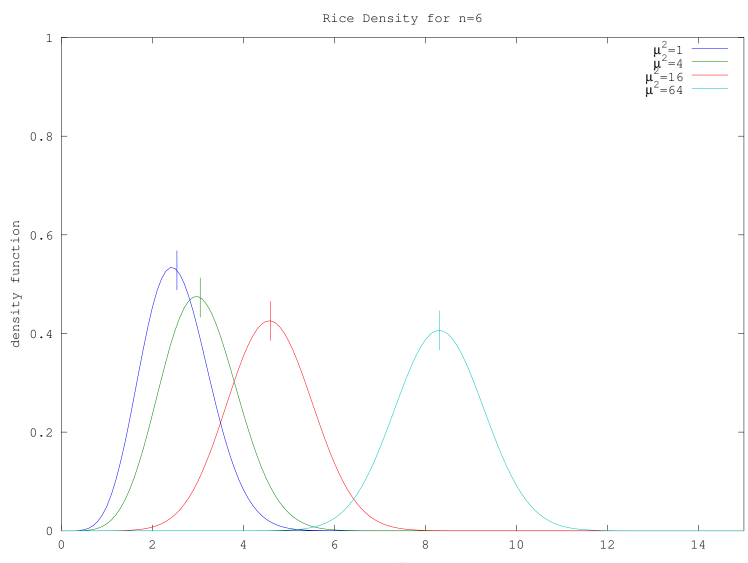

Figure 6: Rice Density for $n = 6$

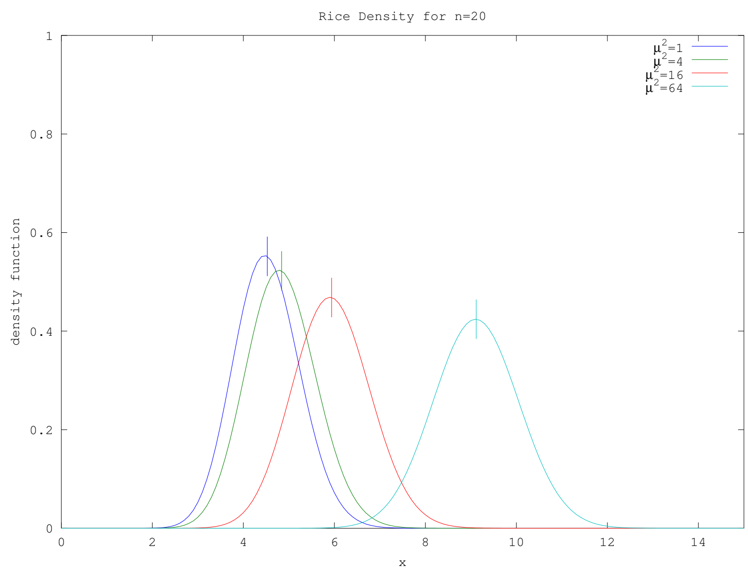

Figure 7: Rice Density for $n = 20$

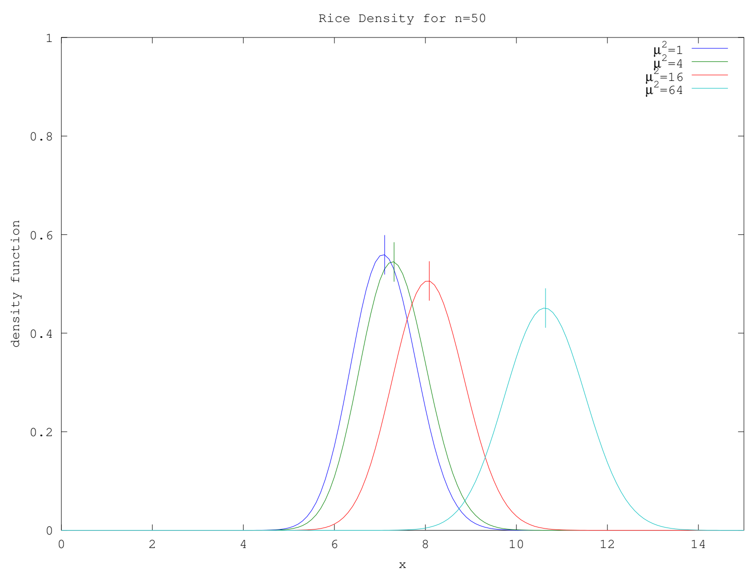

Figure 8: Rice Density for $n = 50$

The Q function for noncentral χ^2 and Marcum's Q function may each be expressed in terms of the other by making simple variable substitutions. Thus

$$\mathrm{Q}_{\bar{\chi}^2}(x; n, m^2) = \mathrm{Q}_{n/2}(\sqrt{2m^2}, \sqrt{2x}) \tag{17}$$

and

$$\mathrm{Q}_{n/2}(\alpha, \beta) = \mathrm{Q}_{\bar{\chi}^2}(\beta^2/2; n, \alpha^2/2). \tag{18}$$

This allows us to express tail probabilities

$$P\{\bar{\chi}^2(n, \mu^2, \sigma^2) > y\} = \mathrm{Q}_{\bar{\chi}^2}(y/2\sigma^2; n, \mu^2/2\sigma^2) = \mathrm{Q}_{n/2}(\sqrt{\mu^2/\sigma^2}, \sqrt{y/\sigma^2}) \tag{19}$$

and

$$P\{\bar{\mathcal{R}}(n, \mu^2, \sigma^2) > y\} = \mathrm{Q}_{n/2}\left(\frac{\mu}{\sigma}, \frac{y}{\sigma}\right) = \mathrm{Q}_{\bar{\chi}^2}(y^2/2\sigma^2; n, \mu^2/2\sigma^2). \tag{20}$$

Clearly Q_M is a probability distribution function and the kernel

$$q_M(x) = \frac{1}{\alpha^{M-1}} x^M e^{-\frac{x^2+\alpha^2}{2}} \mathrm{I}_{M-1}(\alpha x) \chi_{[0,\infty)}. \tag{21}$$

is a probability density function. When n is even, so M is an integer, a number of simplifications occur.

In the following discussion, we will find some of the properties of Bessel functions from Appendix A to be useful. In the remainder of this section we assume that $M = n/2$ is an integer.

3.1 Properties of the Marcum Q Function

In this section we derive some properties of the Marcum Q Function, summarized in Table 1 and Table 2.

Table 1: Properties of the Marcum Q Function

$\mathrm{Q}(\alpha, \beta) = \int_\beta^\infty \nu e^{-\frac{\nu^2+\alpha^2}{2}} \mathrm{I}_0(\alpha\nu) d\nu$
$\mathrm{Q}(\alpha, 0) = 1$
$\mathrm{Q}(0, \beta) = e^{-\frac{\beta^2}{2}}$
$\frac{\partial}{\partial\alpha}\mathrm{Q}(\alpha, \beta) = \beta e^{-\frac{\alpha^2+\beta^2}{2}} \mathrm{I}_1(\alpha\beta)$
$\frac{\partial}{\partial\beta}\mathrm{Q}(\alpha, \beta) = -\beta e^{-\frac{\alpha^2+\beta^2}{2}} \mathrm{I}_0(\alpha\beta)$
$\mathrm{Q}(\alpha, \beta) + \mathrm{Q}(\beta, \alpha) = 1 + e^{-\frac{\alpha^2+\beta^2}{2}} \mathrm{I}_0(\alpha\beta)$
$\mathrm{Q}(\alpha, \beta) = e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{n=0}^\infty \left(\frac{\alpha}{\beta}\right)^n \mathrm{I}_n(\alpha\beta)$
$1 - \mathrm{Q}(\alpha, \beta) = e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{n=1}^\infty \left(\frac{\beta}{\alpha}\right)^n \mathrm{I}_n(\alpha\beta)$

Property 1

$$\mathrm{Q}(\alpha, 0) = 1. \tag{22}$$

Proof: Q is a probability distribution function.

Table 2: Properties of the Generalized Marcum Q Function

$\mathrm{Q}_M(\alpha,\beta) = \frac{1}{\alpha^{M-1}} \int_\beta^\infty \nu^M e^{-\frac{\nu^2+\alpha^2}{2}} \mathrm{I}_{M-1}(\alpha\nu) d\nu$
$\mathrm{Q}_M(\alpha,0) = 1$
$\mathrm{Q}_M(0,\beta) = e^{-\frac{\beta^2}{2}} \sum_{k=0}^{M-1} \frac{\beta^{2k}}{2^k k!}$
$\mathrm{Q}_M(\alpha,\beta) = \mathrm{Q}_{M-1}(\alpha,\beta) + e^{-\frac{\alpha^2+\beta^2}{2}} \left(\frac{\beta}{\alpha}\right)^{M-1} \mathrm{I}_{M-1}(\alpha\beta)$
$\mathrm{Q}_M(\alpha,\beta) = \mathrm{Q}(\alpha,\beta) + e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{k=1}^{M-1} \left(\frac{\beta}{\alpha}\right)^k \mathrm{I}_k(\alpha\beta)$
$\frac{\partial}{\partial\alpha}\mathrm{Q}_M(\alpha,\beta) = \beta e^{-\frac{\alpha^2+\beta^2}{2}} \left(\frac{\beta}{\alpha}\right)^{M-1} \mathrm{I}_M(\alpha\beta)$
$\frac{\partial}{\partial\beta}\mathrm{Q}_M(\alpha,\beta) = -\beta e^{-\frac{\alpha^2+\beta^2}{2}} \left(\frac{\beta}{\alpha}\right)^{M-1} \mathrm{I}_{M-1}(\alpha\beta)$
$Q_M(\alpha,\beta) = e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{k=1-M}^{\infty} \left(\frac{\alpha}{\beta}\right)^k \mathrm{I}_k(\alpha\beta)$
$1 - Q_M(\alpha,\beta) = e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{k=M}^{\infty} \left(\frac{\beta}{\alpha}\right)^k \mathrm{I}_k(\alpha\beta)$
$Q_M(\alpha,\beta) + Q_M(\beta,\alpha) = 1 + e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{k=-(M-1)}^{M-1} \left(\frac{\beta}{\alpha}\right)^k \mathrm{I}_k(\alpha\beta)$

Property 2

$$\mathrm{Q}(0,\beta) = e^{-\frac{\beta^2}{2}}. \tag{23}$$

Proof:

$$\begin{aligned}
\mathrm{Q}(0,\beta) &= \int_\beta^\infty \nu e^{-\frac{\nu^2}{2}} \mathrm{I}_0(0) d\nu \\
&= \int_\beta^\infty \nu e^{-\frac{\nu^2}{2}} d\nu \\
&= e^{-\frac{\beta^2}{2}}.
\end{aligned} \tag{24}$$

Property 3

$$\frac{\partial}{\partial\alpha}\mathrm{Q}(\alpha,\beta) = \beta e^{-\frac{\alpha^2+\beta^2}{2}} \mathrm{I}_1(\alpha\beta). \tag{25}$$

Proof: Using Leibniz' rule,

$$\begin{aligned}
\frac{\partial}{\partial\alpha}\mathrm{Q}(\alpha,\beta) &= \frac{\partial}{\partial\alpha}\int_\beta^\infty \nu e^{-\frac{\nu^2+\alpha^2}{2}} \mathrm{I}_0(\alpha\nu) d\nu \\
&= \int_\beta^\infty \nu \frac{\partial}{\partial\alpha}\left[e^{-\frac{\nu^2+\alpha^2}{2}} \mathrm{I}_0(\alpha\nu)\right] d\nu \\
&= \int_\beta^\infty \nu e^{-\frac{\nu^2+\alpha^2}{2}} \left[\nu \mathrm{I}_1(\alpha\nu) - \alpha \mathrm{I}_0(\alpha\nu)\right] d\nu \\
&= \int_\beta^\infty \nu^2 e^{-\frac{\nu^2+\alpha^2}{2}} \mathrm{I}_1(\alpha\nu) d\nu - \alpha\mathrm{Q}(\alpha,\beta).
\end{aligned} \tag{26}$$

Integrating by parts with $dv = \nu e^{-\frac{\alpha^2+\nu^2}{2}} d\nu$ *and* $u = \nu \mathrm{I}_1(\alpha\nu)$ *we obtain*

$$\begin{aligned}
\int_\beta^\infty \nu^2 e^{-\frac{\nu^2+\alpha^2}{2}} \mathrm{I}_1(\alpha\nu) d\nu &= -\nu e^{-\frac{\nu^2+\alpha^2}{2}} \mathrm{I}_1(\alpha\nu)\Big|_\beta^\infty + \alpha \int_\beta^\infty \nu^2 e^{-\frac{\nu^2+\alpha^2}{2}} \mathrm{I}_0(\alpha\nu) d\nu \\
&= \beta e^{-\frac{\alpha^2+\beta^2}{2}} \mathrm{I}_1(\alpha\beta) + \alpha\mathrm{Q}(\alpha,\beta).
\end{aligned} \tag{27}$$

Property 4

$$\frac{\partial}{\partial\beta}\mathrm{Q}(\alpha,\beta) = -\beta e^{-\frac{\alpha^2+\beta^2}{2}}\mathrm{I}_0(\alpha\beta). \tag{28}$$

Proof: Easily follows from Leibniz' rule.

Property 5

$$\mathrm{Q}(\alpha,\beta) + \mathrm{Q}(\beta,\alpha) = 1 + e^{-\frac{\alpha^2+\beta^2}{2}}\mathrm{I}_0(\alpha,\beta). \tag{29}$$

Proof: Differentiating and using the above properties

$$\frac{\partial}{\partial\alpha}\left[\mathrm{Q}(\alpha,\beta) + \mathrm{Q}(\beta,\alpha)\right] = \beta e^{-\frac{\alpha^2+\beta^2}{2}}\mathrm{I}_1(\alpha,\beta) - \alpha e^{-\frac{\alpha^2+\beta^2}{2}}\mathrm{I}_0(\alpha,\beta) = \frac{\partial}{\partial\alpha}\left[e^{-\frac{\alpha^2+\beta^2}{2}}\mathrm{I}_0(\alpha,\beta)\right]. \tag{30}$$

Integrating we have

$$\mathrm{Q}(\alpha,\beta) + \mathrm{Q}(\beta,\alpha) = e^{-\frac{\alpha^2+\beta^2}{2}}\mathrm{I}_0(\alpha,\beta) + \mathrm{f}(\beta) \tag{31}$$

where f *is an arbitrary function of* β. *But*

$$\mathrm{Q}(0,\beta) + \mathrm{Q}(\beta,0) = 1 + e^{-\frac{\beta^2}{2}} \tag{32}$$

so $\mathrm{f}(\beta) = 1$.

Property 6

$$1 - \mathrm{Q}(\alpha,\beta) = e^{-\frac{\alpha^2+\beta^2}{2}}\sum_{n=1}^{N}\left(\frac{\beta}{\alpha}\right)^n\mathrm{I}_n(\alpha\beta) + \frac{1}{\alpha^N}\int_0^\beta \nu^{N+1}e^{-\frac{\alpha^2+\nu^2}{2}}\mathrm{I}_N(\alpha\nu)d\nu. \tag{33}$$

Proof: By definition

$$1 - \mathrm{Q}(\alpha,\beta) = \int_0^\beta \nu e^{-\frac{\nu^2+\alpha^2}{2}}\mathrm{I}_0(0)d\nu. \tag{34}$$

If we set

$$u = e^{-\frac{\nu^2+\alpha^2}{2}} \tag{35}$$

and

$$dv = \nu\mathrm{I}_0(\alpha\nu)d\nu \tag{36}$$

using Equation 117 we have

$$1 - \mathrm{Q}(\alpha,\beta) = e^{-\frac{\alpha^2+\beta^2}{2}}\left(\frac{\beta}{\alpha}\right)\mathrm{I}_1(\alpha\beta) + \frac{1}{\alpha}\int_0^\beta \nu^2 e^{-\frac{\alpha^2+\nu^2}{2}}\mathrm{I}_1(\alpha\nu)d\nu. \tag{37}$$

Thus the property is true for $N = 1$. *Assume that*

$$1 - \mathrm{Q}(\alpha,\beta) = e^{-\frac{\alpha^2+\beta^2}{2}}\sum_{n=1}^{N-1}\left(\frac{\beta}{\alpha}\right)^n\mathrm{I}_n(\alpha\beta) + \frac{1}{\alpha^{N-1}}\int_0^\beta \nu^N e^{-\frac{\alpha^2+\nu^2}{2}}\mathrm{I}_{N-1}(\alpha\nu)d\nu. \tag{38}$$

Integrating by parts again with

$$u = e^{-\frac{\nu^2+\alpha^2}{2}} \tag{39}$$

and

$$dv = \nu^N\mathrm{I}_{N-1}(\alpha\nu)d\nu \tag{40}$$

We have the result for N *and the proof is complete. Note that we really need to deal with convergence issues. We will blow that off here, but techniques to deal with the problem may be found in various places including [6].*

Property 7

$$1 - \mathrm{Q}(\alpha, \beta) = e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{n=1}^{\infty} \left(\frac{\beta}{\alpha}\right)^n \mathrm{I}_n(\alpha\beta) \tag{41}$$

Proof: This follows directly from Property 6.

Property 8

$$\mathrm{Q}(\alpha, \beta) = e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{n=0}^{\infty} \left(\frac{\alpha}{\beta}\right)^n \mathrm{I}_n(\alpha\beta) \tag{42}$$

Proof: Using the properties already discussed

$$\mathrm{Q}(\alpha, \beta) = e^{-\frac{\alpha^2+\beta^2}{2}} \mathrm{I}_0(\alpha, \beta) + (1 - \mathrm{Q}(\beta, \alpha)) = e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{n=0}^{\infty} \left(\frac{\alpha}{\beta}\right)^n \mathrm{I}_n(\alpha\beta). \tag{43}$$

Property 9

$$\mathrm{Q}_M(\alpha, \beta) = \mathrm{Q}_{M-1}(\alpha, \beta) + e^{-\frac{\alpha^2+\beta^2}{2}} \left(\frac{\beta}{\alpha}\right)^{M-1} \mathrm{I}_{M-1}(\alpha\beta) \tag{44}$$

Proof: Integration by parts.

Property 10

$$\mathrm{Q}_M(\alpha, \beta) = \mathrm{Q}(\alpha, \beta) + e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{m=1}^{M-1} \left(\frac{\beta}{\alpha}\right)^m \mathrm{I}_m(\alpha\beta) \tag{45}$$

Proof: Repeated application of the previous property.

Property 11

$$\frac{\partial}{\partial\alpha} \mathrm{Q}_M(\alpha, \beta) = \beta e^{-\frac{\alpha^2+\beta^2}{2}} \left(\frac{\beta}{\alpha}\right)^{M-1} \mathrm{I}_M(\alpha\beta) \tag{46}$$

Proof: Differentiate

$$\mathrm{Q}_M(\alpha, \beta) = \mathrm{Q}(\alpha, \beta) + e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{m=1}^{M-1} \left(\frac{\beta}{\alpha}\right)^m \mathrm{I}_m(\alpha\beta) \tag{47}$$

and combine terms.

Property 12

$$\frac{\partial}{\partial\beta} \mathrm{Q}_M(\alpha, \beta) = -\beta e^{-\frac{\alpha^2+\beta^2}{2}} \left(\frac{\beta}{\alpha}\right)^{M-1} \mathrm{I}_{M-1}(\alpha\beta) \tag{48}$$

Proof: Differentiate

$$\mathrm{Q}_M(\alpha, \beta) = \mathrm{Q}(\alpha, \beta) + e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{m=1}^{M-1} \left(\frac{\beta}{\alpha}\right)^m \mathrm{I}_m(\alpha\beta) \tag{49}$$

and combine terms.

Property 13

$$\mathrm{Q}_M(\alpha, 0) = 1. \tag{50}$$

Proof: Q_M *is a probability distribution.*

Property 14

$$Q_M(0,\beta) = e^{-\frac{\beta^2}{2}} \sum_{k=0}^{M-1} \frac{\beta^{2k}}{2^k k!} \tag{51}$$

Proof: This follows from the value of $Q(0,\beta)$ and Property 10. Note that when $\alpha = 0$ we have a χ^2 variable and this result is the same.

Property 15

$$Q_M(\alpha,\beta) + Q_M(\beta,\alpha) = 1 + e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{k=-(M-1)}^{M-1} \left(\frac{\beta}{\alpha}\right)^k \mathrm{I}_k(\alpha\beta). \tag{52}$$

Proof: Combine Property 5 and Property 10.

Property 16 *The generalized Marcum Q function may be written (for $\alpha > 0$ and $\beta > 0$)*

$$Q_M(\alpha,\beta) = \begin{cases} e^{-\frac{(\alpha-\beta)^2}{2}} \left(\frac{\beta}{\alpha}\right)^M \int_0^{2\pi} e^{\alpha\beta(\cos\theta-1)} \frac{\frac{\beta}{\alpha}\cos(M-1)\theta - \cos M\theta}{\left(\frac{\beta}{\alpha}\right)^2 - 2\frac{\beta}{\alpha}\cos\theta + 1} \frac{d\theta}{2\pi} & \alpha < \beta \\ \frac{1}{2} + \frac{1}{2}\int_0^{2\pi} e^{\alpha^2(\cos\theta-1)} \frac{\sin\left(M-\frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta} \frac{d\theta}{2\pi} & \alpha = \beta \\ 1 + e^{-\frac{(\alpha-\beta)^2}{2}} \left(\frac{\beta}{\alpha}\right)^M \int_0^{2\pi} e^{\alpha\beta(\cos\theta-1)} \frac{\frac{\beta}{\alpha}\cos(M-1)\theta - \cos M\theta}{\left(\frac{\beta}{\alpha}\right)^2 - 2\frac{\beta}{\alpha}\cos\theta + 1} \frac{d\theta}{2\pi} & \alpha > \beta \end{cases} . \tag{53}$$

Proof: First, for $\alpha < \beta$, we have

$$\begin{aligned} Q_M(\alpha,\beta) &= e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{k=1-M}^{\infty} \left(\frac{\alpha}{\beta}\right)^k \mathrm{I}_k(\alpha\beta) \\ &= e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{k=1-M}^{\infty} \left(\frac{\alpha}{\beta}\right)^k \int_0^{2\pi} \cos k\theta e^{\alpha\beta\cos\theta} \frac{d\theta}{2\pi} \\ &= e^{-\frac{\alpha^2+\beta^2}{2}} \frac{1}{2} \int_0^{2\pi} e^{\alpha\beta\cos\theta} \sum_{k=1-M}^{\infty} \left(\frac{\alpha}{\beta}\right)^k \left(e^{ik\theta} + e^{-ik\theta}\right) \frac{d\theta}{2\pi} \\ &= e^{-\frac{\alpha^2+\beta^2}{2}} \frac{1}{2} \int_0^{2\pi} e^{\alpha\beta\cos\theta} \left(\frac{\alpha}{\beta}\right)^{1-M} \left(\frac{e^{i(1-M)\theta}}{1-\frac{\alpha}{\beta}e^{i\theta}} + \frac{e^{-i(1-M)\theta}}{1-\frac{\alpha}{\beta}e^{-i\theta}}\right) \frac{d\theta}{2\pi} \\ &= e^{-\frac{(\alpha-\beta)^2}{2}} \left(\frac{\alpha}{\beta}\right)^{1-M} \int_0^{2\pi} e^{\alpha\beta(\cos\theta-1)} \frac{\cos(M-1)\theta - \frac{\alpha}{\beta}\cos M\theta}{1 - 2\frac{\alpha}{\beta}\cos\theta + \left(\frac{\alpha}{\beta}\right)^2} \frac{d\theta}{2\pi}. \\ &= e^{-\frac{(\alpha-\beta)^2}{2}} \left(\frac{\beta}{\alpha}\right)^M \int_0^{2\pi} e^{\alpha\beta(\cos\theta-1)} \frac{\frac{\beta}{\alpha}\cos(M-1)\theta - \cos M\theta}{\left(\frac{\beta}{\alpha}\right)^2 - 2\frac{\beta}{\alpha}\cos\theta + 1} \frac{d\theta}{2\pi}. \end{aligned} \tag{54}$$

Note that the sum in the third step converges if and only if $\alpha < \beta$.

If $\alpha > \beta$, we have

$$\begin{aligned} 1 - Q_M(\alpha,\beta) &= e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{k=M}^{\infty} \left(\frac{\beta}{\alpha}\right)^k \mathrm{I}_k(\alpha\beta) \\ &= e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{k=M}^{\infty} \left(\frac{\beta}{\alpha}\right)^k \int_0^{2\pi} \cos k\theta e^{\alpha\beta\cos\theta} \frac{d\theta}{2\pi} \end{aligned}$$

$$
\begin{aligned}
&= e^{-\frac{\alpha^2+\beta^2}{2}} \frac{1}{2} \int_0^{2\pi} e^{\alpha\beta\cos\theta} \sum_{k=M}^{\infty} \left(\frac{\beta}{\alpha}\right)^k \left(e^{ik\theta} + e^{-ik\theta}\right) \frac{d\theta}{2\pi} \\
&= e^{-\frac{\alpha^2+\beta^2}{2}} \frac{1}{2} \int_0^{2\pi} e^{\alpha\beta\cos\theta} \left(\frac{\beta}{\alpha}\right)^M \left(\frac{e^{iM\theta}}{1-\frac{\beta}{\alpha}e^{i\theta}} + \frac{e^{-iM\theta}}{1-\frac{\beta}{\alpha}e^{-i\theta}}\right) \frac{d\theta}{2\pi} \\
&= e^{-\frac{(\alpha-\beta)^2}{2}} \left(\frac{\beta}{\alpha}\right)^M \int_0^{2\pi} e^{\alpha\beta(\cos\theta-1)} \frac{\cos M\theta - \frac{\beta}{\alpha}\cos(M-1)\theta}{\left(\frac{\beta}{\alpha}\right)^2 - 2\frac{\beta}{\alpha}\cos\theta + 1} \frac{d\theta}{2\pi}. \qquad (55)
\end{aligned}
$$

Finally, if $\alpha = \beta$, we have

$$
\begin{aligned}
2\mathrm{Q}(\alpha,\alpha) &= 1 + e^{-\frac{\alpha^2}{2}} \sum_{k=-(M-1)}^{M-1} I_k(\alpha^2) \\
&= 1 + e^{-\frac{\alpha^2}{2}} \int_0^{2\pi} e^{\alpha^2\cos\theta} \frac{1}{2} \sum_{k=-(M-1)}^{M-1} \left(e^{ik\theta} + e^{-ik\theta}\right) \frac{d\theta}{2\pi} \\
&= 1 + \int_0^{2\pi} e^{\alpha^2(\cos\theta-1)} \left(\frac{e^{-i(M-1)\theta} - e^{-iM\theta}}{1-e^{i\theta}}\right) \frac{d\theta}{2\pi} \\
&= 1 + \int_0^{2\pi} e^{\alpha^2(\cos\theta-1)} \frac{\sin(M-\frac{1}{2})\theta}{\sin\frac{\theta}{2}} \frac{d\theta}{2\pi}. \qquad (56)
\end{aligned}
$$

4 Computing the Inverse of Marcum's Q Function

There are many situations in which, given a probability and Rice distribution parameters, it is desirable to find the threshold at which the probability occurs. In other words, given α and a probability p, we would like to find the value of β that solves

$$p = \mathrm{Q}_M(\alpha, \beta). \qquad (57)$$

To the best of the author's knowledge there is no direct solution to the inverse problem, but we are able to compute the Q function and we know the derivative of the Q function (see Section 3) suggesting the use of a Newton-Raphson iteration.

The Newton-Raphson iteration[7] begins with an initial estimate β_0 and computes an updated estimates

$$\beta_{k+1} = \beta_k - \frac{\mathrm{Q}_M(\alpha,\beta_k) - p}{\frac{\partial}{\partial\beta}\mathrm{Q}_M(\alpha,\beta_k)} \qquad (58)$$

where

$$\frac{\partial}{\partial\beta}\mathrm{Q}_M(\alpha,\beta) = -\beta e^{-\frac{\alpha^2+\beta^2}{2}} \left(\frac{\beta}{\alpha}\right)^{M-1} \mathrm{I}_{M-1}(\alpha\beta). \qquad (59)$$

Unfortunately the iteration normally diverges, but if we work with the log of the Q function, convergence is essentially guaranteed if a reasonable initial estimate is chosen. Thus the iteration becomes

$$\beta_{k+1} = \beta_k - Q_M(\alpha,\beta_k) \frac{\log\left(\mathrm{Q}_M(\alpha,\beta_k)\right) - \log p}{\frac{\partial}{\partial\beta}\mathrm{Q}_M(\alpha,\beta_k)} \qquad (60)$$

and terminates when the update is small enough. For the code supplied on the website, the distribution mean is chosen as the initial estimate. This is a very good choice for all reasonable choices of the parameters but does have some unfortunate issues as described in the following paragraphs.

A discussion of some cases where the algorithm apparently fails is worthwhile. In the following, `marcumQ` is an implementation of any of the algorithms described in the sequel and `imarcumQ` is the inverse implemented using `marcumQ` and the Newton-Raphson iteration described above.

For the first example suppose we have $M = 5$, $\alpha = 5$, and we compute $Q_5(5, 0) = 1$ so that $p = 1$. When we compute the inverse we obtain (the following is obtained by running the octave scripts)

```
octave> alpha = 5; beta=0; M=5;
octave> p = marcumQ(alpha,beta,M)
p = 1
octave> betahat = imarcumQ(p, alpha, M)
betahat = 0.186411
```

so that the estimate of the inverse is clearly not what we started with. However if we compute

```
octave> phat = marcumQ(alpha,betahat,M)
phat = 1
```

so that, to machine precision, the inverse is correct. The Q function is very flat near zero and for large β, so the inverse is mathematically unique but numerically ambiguous in those regions.

A similar example is attempting to compute the inverse when the Q function is extremely small. Consider

```
octave> alpha = 5; beta=14; M=5;
octave> p = marcumQ(alpha,beta,M)
p = 1.07456e-17
octave> betahat = imarcumQ(p, alpha, M)
betahat = 50.31195
```

so again the estimate of the inverse is not what we expected. However

```
octave> phat = marcumQ(alpha,betahat,M)
phat = 0
```

which well within the available tolerance for the algorithm. As a side note, a better choice of initial estimate would have yielded a better estimate of the inverse, but the algorithm with the Bessel functions used in `octave` probably do not yield usable results down to 10^{-17} anyway.

The examples shown demonstrate that, while the inverse algorithm is quite robust, an understanding of the numerical nature of the problem is essential to interpreting the results.

5 Computation of Marcum's Q Function

In this section we present various algorithms used for the computation of probabilities for Rice and noncentral χ^2 random variables. In all cases we assume that the number of degrees of freedom is an even integer, and all of the algorithms will be put in the form of the Marcum Q function.

5.1 The Bessel Series

One method for computing the Marcum Q function is to use the series expansions developed in Section 3.1, repeated here for convenience

$$Q_M(\alpha,\beta) = e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{k=1-M}^{\infty} \left(\frac{\alpha}{\beta}\right)^k \mathrm{I}_k(\alpha\beta) \tag{61}$$

and

$$1 - Q_M(\alpha,\beta) = e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{k=M}^{\infty} \left(\frac{\beta}{\alpha}\right)^k \mathrm{I}_k(\alpha\beta). \tag{62}$$

Convergence is generally more rapid if we use the first form when $\alpha < \beta$ and the second form otherwise, although for large M it might be be better to just use the second form regardless.

Numerically this series has the problem that I_k grows exponentially with $\alpha\beta$, and the exponential weighting term decays exponentially. If we make the expansion

$$e^{-\frac{\alpha^2+\beta^2}{2}} \mathrm{I}_k(\alpha\beta) = e^{-\frac{(\alpha-\beta)^2}{2}} \left(e^{-\alpha\beta}\mathrm{I}_k(\alpha\beta)\right) \tag{63}$$

then the problem is reduced significantly. Furthermore some Bessel function software computes

$$e^{-z}\mathrm{I}_k(z) \tag{64}$$

and then optionally removes the exponential term.

The performance of this technique is entirely dependant on the accuracy and speed of the Bessel function computation. The routine used in `octave` comes from [8] and is fairly fast, accurate, and efficient for all arguments and orders.

5.2 Computation of the Marcum Q Function Using Parl's Method

Parl's method ([4]) is based on the backward-recursion method for computing modified Bessel functions. He suggests using the series expansion

$$Q_M(\alpha,\beta) = e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{k=1-M}^{\infty} \left(\frac{\alpha}{\beta}\right)^k \mathrm{I}_k(\alpha\beta) \tag{65}$$

when $\alpha < \beta$ and

$$1 - Q_M(\alpha,\beta) = e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{k=M}^{\infty} \left(\frac{\beta}{\alpha}\right)^k \mathrm{I}_k(\alpha\beta) \tag{66}$$

when $\alpha \geq \beta$. Both of these functions may be written in the form (recognizing that $\mathrm{I}_k = \mathrm{I}_{-k}$ for integer values of k)

$$e^{-\frac{(\alpha-\beta)^2}{2}} \sum_{k=0}^{\infty} F_k e^{-\alpha\beta}\mathrm{I}_k(\alpha\beta) = e^{-\frac{(\alpha-\beta)^2}{2}} e^{-\alpha\beta} F(\alpha\beta) \tag{67}$$

where

$$F_k = \begin{cases} \left(\frac{\alpha}{\beta}\right)^k, & k \geq M, \\ \left(\frac{\alpha}{\beta}\right)^k + \left(\frac{\beta}{\alpha}\right)^k, & 1 \leq k < M, \\ 1, & k = 0, \end{cases} \tag{68}$$

for the computation of Q_M and

$$F_k = \begin{cases} \left(\frac{\beta}{\alpha}\right)^k, & k \geq M, \\ 0, & k < M, \end{cases} \tag{69}$$

for the computation of $1 - \mathrm{Q}_M$. Similarly, since

$$e^z = \mathrm{I}_0(z) + 2\sum_{k=1}^{\infty} \mathrm{I}_k(z) \tag{70}$$

we may write

$$e^z = \sum_{k=0}^{\infty} G_k \mathrm{I}_k(z) = G(z) \tag{71}$$

with

$$G_k = \begin{cases} 1, & k = 0, \\ 2, & k > 0, \end{cases} . \tag{72}$$

It is well understood that the Bessel recursion

$$\mathrm{I}_{k-1}(z) = \mathrm{I}_{k+1}(z) + \frac{2k}{z}\mathrm{I}_k(z) \tag{73}$$

may be used to compute accurate estimates of the modified Bessel functions I_k. Let

$$g_k(z;N) = \begin{cases} 0, & k > N, \\ 1, & k = N, \\ g_{k+2}(z;N) + \frac{2(k+1)}{z} g_{k+1}(z;N), & 0 \leq k < N, \end{cases} . \tag{74}$$

then $g_k(z;N) \approx \gamma \mathrm{I}_k(z)$ for some scale factor γ and k much smaller than N. The scale factor is found by summing the g_k.

The backward recursion relies on picking N large enough so that errors damp out, but Parl's method is a clever way of updating the backward recursion in a forward way. We start with $N = 0$ and update the backward recursion until the errors are acceptable. For this we need the order-update recursion (with the proof at the end of this section)

$$g_k(z;N) = \frac{2N}{z} g_k(z;N-1) + g_k(z;N-2) \tag{75}$$

provided $k < N$.

Generalizing Equations 67 and 71, define

$$\hat{P}(z;N) = \sum_{k=0}^{N} P_k g_k(z;N) \tag{76}$$

for some sequence of P_k, then we may build this using the backward recursion and the order-update recursion as (proof at the end of this section)

$$\hat{P}(z;N) = P_N + \frac{2N}{z}\hat{P}(z;N-1) + \hat{P}(z;N-2) \tag{77}$$

with $\hat{P}(z;-1) = 0$ and $\hat{P}(z;0) = P_0$.

Returning to the problem of computing Q_M or $1 - \mathrm{Q}_M$, we have

$$F(z) = \sum_{k=0}^{\infty} F_k \mathrm{I}_k(z) \approx \gamma \sum_{k=0}^{N} F_k g_k(z; N) \doteq \hat{F}(z; N) \tag{78}$$

and

$$\gamma e^z = \sum_{k=0}^{\infty} G_k \mathrm{I}_k(z) \approx \gamma \sum_{k=0}^{N} G_k g_k(z; N) \doteq \hat{G}(z; N) \tag{79}$$

so the function we are attempting to compute is approximately

$$e^{-\frac{\alpha^2+\beta^2}{2}} \frac{\hat{F}(z; N)}{\hat{G}(z; N)}. \tag{80}$$

We use the forward update equations for both $\hat{F}$ and $\hat{G}$ and stop when we are close enough.

Deciding when we are close enough is an issue. Both the numerator and denominator may grow very large, and Parl suggests terminating the recursion when the numerator and denominator exceed a bound that is determined from the arguments and desired tolerance. A simpler approach, made at a very small expense in computational complexity is to terminate when the algorithm asymptotes. In particular if

$$\gamma_N = \frac{\hat{F}(z; N)}{\hat{G}(z; N)} \tag{81}$$

then we terminate when

$$|1 - \gamma_N / \gamma_{N-1}| < \Delta \tag{82}$$

where Δ is the desired tolerance.

Parl's algorithm with the above termination condition is summarized in Table 3.

A general statement about Parl's paper. Parl's method is delightfully clever, but it almost seems as if the paper was written for maximum obfuscation. Complex details were dismissed with statements like "after some manipulation". In general the paper was written in such a way as to be horrendously difficult to read and leaves out critical details. Those details are included above.

Table 3: Parl's Iteration

	Coefficients	Initial	Iteration	Quit	Final
$\alpha < \beta$	Equation 68	$\hat{F}_{-1} = 0$ $\hat{F}_0 = F_0$	$\hat{F}_N = F_N + \frac{2N}{z}\hat{F}_{N-1} + \hat{F}_{N-2}$	$\epsilon_N < \Delta$	$e^{-\frac{(\alpha-\beta)^2}{2}} \frac{\hat{F}_N}{\hat{G}_N}$
$\alpha \geq \beta$	Equation 69	$\hat{G}_{-1} = 0$ $\hat{G}_0 = 1$	$\hat{G}_N = 2 + \frac{2N}{z}\hat{G}_{N-1} + \hat{G}_{N-2}$ $\gamma_N = \hat{F}_N / \hat{G}_N$ $\epsilon_N = \lvert 1 - \gamma_N/\gamma_{N-1} \rvert$		$1 - e^{-\frac{(\alpha-\beta)^2}{2}} \frac{\hat{F}_N}{\hat{G}_N}$

5.2.1 Formal Theorems and Proofs for Parl's Method

Theorem 1

$$g_k(z; N) = \frac{2N}{z} g_k(z; N-1) + g_k(z; N-2). \tag{83}$$

Proof: First we will verify the recursion for small N. For $N = 1$ we can apply either the backward recursion or the order-update recursion to obtain

$$g_1(z;1) = 1, g_0(z;1) = \frac{2}{z}. \tag{84}$$

Similarly for $N = 2$, we have

$$g_2(z;2) = 1, g_1(z;2) = \frac{4}{z}, g_0(z;2) = 1 + \frac{8}{z^2} \tag{85}$$

by applying either the backward recursion or the order-update recursion.

For general N, we start with the backward recursion

$$g_N(z;N) = g_{N-1}(z;N-1) = 1, \tag{86}$$

$$g_{N+1}(z;N) = g_N(z;N-1) = 0, \tag{87}$$

$$g_{N-1}(z;N) = \frac{2N}{z} g_N(z;N) + g_{N+1}(z;N) = \frac{2N}{z} g_{N-1}(z;N-1) + g_{N-1}(z;N-2). \tag{88}$$

Similarly

$$\begin{aligned} g_{N-2}(z;N) &= \frac{2(N-1)}{z} g_{N-1}(z;N) + g_N(z;N) \\ &= \frac{2(N-1)}{z}\left(\frac{2N}{z}\right) + 1 \\ &= \frac{2N}{z}\left(\frac{2(N-1)}{z}\right) + 1 \\ &= \frac{2N}{z} g_{N-2}(z;N-1) + g_{N-2}(z;N-2). \end{aligned} \tag{89}$$

Now assume that

$$g_n(z;N) = \frac{2N}{z} g_n(z;N-1) + g_n(z;N-2) \tag{90}$$

holds for $n > k$ and compute

$$\begin{aligned} g_k(z;N) &= \frac{2(k+1)}{z} g_{k+1}(z;N) + g_{k+2}(z;N) \\ &= \frac{2(k+1)}{z}\left(\frac{2N}{z} g_{k+1}(z;N-1) + g_{k+1}(z;N-2)\right) \\ &+ \left(\frac{2N}{z} g_{k+2}(z;N-1) + g_{k+2}(z;N-2)\right) \\ &= \frac{2N}{z}\left(\frac{2(k+1)}{z} g_{k+1}(z;N-1) + g_{k+2}(z;N-1)\right) \\ &+ \left(\frac{2(k+1)}{z} g_{k+1}(z;N-2) + g_{k+2}(z;N-2)\right) \\ &= \frac{2N}{z} g_k(z;N-1) + g_k(z;N-2) \end{aligned} \tag{91}$$

and the proof is complete.

Theorem 2 *If*

$$\hat{P}(z;N) = \sum_{k=0}^{N} P_k g_k(z;N) \tag{92}$$

then

$$\hat{P}(z;N) = P_N + \frac{2N}{z}\hat{P}(z;N-1) + \hat{P}(z;N-2). \tag{93}$$

Proof:

$$\begin{aligned} \hat{P}(z;N) &= \sum_{k=0}^{N} P_k g_k(z;N) \\ &= P_N + \sum_{k=0}^{N-1} P_k \left(\frac{2N}{z} g_k(z;N-1) + g_k(z;N-2)\right) \\ &= P_N + \frac{2N}{z}\hat{P}(z;N-1) + \hat{P}(z;N-2). \end{aligned} \tag{94}$$

5.3 Computation of the Noncentral χ^2 Q Function Using Dillard's Algorithm

Dillard's algorithm (more accurately Dillard's extension of McGee's algorithm) is simply a computation of a series expansion of the $\mathrm{Q}_{\bar{\chi}^2}$ function, made more efficient by reusing computations. Following the development in [5] we start with

$$\mathrm{Q}_{\bar{\chi}^2}(x;n,m^2) = \int_x^\infty (\sqrt{w})^{n/2-1} e^{-(m^2+w)} \frac{\mathrm{I}_{n/2-1}\left(2\sqrt{m^2 w}\right)}{m^{n/2-1}} dw. \tag{95}$$

The modified Bessel functions may be expanded in a series

$$\mathrm{I}_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}z\right)^{2k}}{k!\Gamma(\nu+k+1)}. \tag{96}$$

Thus

$$\begin{aligned} \mathrm{Q}_{\bar{\chi}^2}(x;n,m^2) &= \frac{e^{-m^2}}{m^{n/2-1}} \int_x^\infty (\sqrt{w})^{n/2-1} e^{-w} \left(\sqrt{m^2 w}\right)^{(n/2-1)} \sum_{n=0}^{\infty} \frac{\left(m^2 w\right)^k}{k!\Gamma(n/2+k)} dw. \\ &= e^{-m^2} \sum_{n=0}^{\infty} \frac{m^{2k}}{\Gamma(n/2+k)} \frac{1}{k!} \int_x^\infty w^{n/2+k-1} e^{-w} dw. \end{aligned} \tag{97}$$

If we assume n is an even integer and define $M = n/2$ then

$$\mathrm{Q}_{\bar{\chi}^2}(x;n,m^2) = \sum_{n=0}^{\infty} \frac{m^{2k}e^{-m^2}}{k!} \frac{1}{(M+k-1)!} \int_x^\infty w^{M+k-1} e^{-w} dw. \tag{98}$$

If we let

$$b_k = \frac{m^{2k}}{k!} e^{-m^2} \tag{99}$$

and

$$g_k = \frac{1}{k!} \int_x^\infty w^k e^{-w} dw \tag{100}$$

then

$$\mathrm{Q}_{\bar{\chi}^2}(x; n, m^2) = \sum_{n=0}^{\infty} b_k g_{M+k-1} \tag{101}$$

and we can create a simple recursion for the sum. Trivially, since $b_0 = e^{-m^2}$,

$$b_{k+1} = \frac{m^2}{k+1} b_k. \tag{102}$$

We use integration by parts to find the recursion for g_k For $k > 0$,

$$\begin{aligned} g_{k+1} &= \frac{1}{(k+1)!}\left[x^{k+1}e^{-x} + (k+1)\int_x^\infty w^k e^{-w} dw\right] \\ &= \frac{x^{k+1}e^{-x}}{(k+1)!} + \frac{1}{k!}\int_x^\infty w^k e^{-w} dw \\ &= \frac{x^{k+1}e^{-x}}{(k+1)!} + g_k. \end{aligned} \tag{103}$$

If we set

$$g_0 = e^{-x} \tag{104}$$

and

$$\delta_k = \frac{x^k e^{-x}}{k!} \tag{105}$$

then, for $k \geq 0$, the recursion becomes

$$\delta_{k+1} = \frac{x}{k+1}\delta_k. \tag{106}$$

and

$$g_{k+1} = \delta_{k+1} + g_k. \tag{107}$$

We also find it useful to compute

$$1 - \mathrm{Q}_{\bar{\chi}^2}(x; n, m^2) = \int_0^x (\sqrt{w})^{n/2-1} e^{-(m^2+w)} \frac{\mathrm{I}_{n/2-1}\left(2\sqrt{m^2 w}\right)}{m^{n/2-1}} dw. \tag{108}$$

Following the same development, we have $g_0 = 1 - e^{-x}$ and

$$g_{k+1} = -\delta_k + g_k. \tag{109}$$

with the iteration for δ_k the same as before. The algorithm is summarized in Tables 4 and 5.

The author's code uses Dillard's algorithm to compute the generalized Marcum Q function $\mathrm{Q}_M(\alpha, \beta)$ using the transformations described in Section 3.1. Furthermore in [5] it is suggested that Q_M be computed when $\beta > E\bar{\mathcal{R}}(n, \mu^2, \sigma^2)$ and $1 - \mathrm{Q}_M$ otherwise. The author's code uses the nearly identical expedient of computing Q_M with $\alpha > \beta$ and $1 - Q_M$ otherwise.

Table 4: The g-δ Iteration

	Initial Conditions		Iteration ($k \geq 0$)	
Q	$g_0 = e^{-x}$	$\delta_0 = e^{-x}$	$\delta_{k+1} = \frac{x}{k+1}\delta_k$	$g_{k+1} = \delta_{k+1} + g_k$
$1 - \mathrm{Q}$	$g_0 = 1 - e^{-x}$			$g_{k+1} = -\delta_{k+1} + g_k$

Table 5: Iteration For Computing $\mathrm{Q}_M(\alpha, \beta)$ Function

Initial Conditions	Iteration ($k \geq 0$)	Termination
Perform the g-δ iteration $M - 1$ times $x = \beta^2/2$ $m^2 = \alpha^2/2$ $b_0 = e^{-m^2}$ $S_0 = 0$	$S_{k+1} = S_k + b_k g_{M+k-1}$ $b_{k+1} = \frac{m^2}{k+1} b_k$ compute δ_{M+k},g_{M+k}	$\left\lvert \frac{b_k g_{M+k-1}}{S_{k+1}} \right\rvert < \Delta$

6 Comparison and Performance

In this section we present some performance results. The results presented here are all using `octave`.

For each of the algorithms presented here we present two sets of charts for a variety of orders and noncentrality parameters. The first set of charts shows the number of iterations required to meet the tolerance conditions for the algorithm in question. The second set of charts shows the actual CPU time for the same set of parameters. Of course CPU time is dependent on the computer, so the results are normalized to give a convenient scale on the axes.

Each plot has three vertical hashmarks per trace. The leftmost hash is the point at which the Q function has the value $1 - 10^{-6}$. The center hash is the mean of the distribution. The rightmost hash is the point at which the Q function has the value 10^{-6}.

As the reader peruses the charts in the following section, it will become clear that there is no clear “winner” in the performance games. However, in general the author computes probabilities in the tails, that is for $\alpha < \beta$. In this case the series method is faster than the other methods and so is the method the author has placed in his own library. Most likely the other methods provide better accuracy, but when a large number of terms are used in the computation a significant roundoff error is likely to accumulate. The author has not yet made a serious effort at estimating roundoff error.

6.1 The Series Approach

The most straightforward approach to computing Marcum’s Q function is to use the series forms as described in an earlier section.

$$Q_M(\alpha, \beta) = e^{-\frac{(\alpha-\beta)^2}{2}} \sum_{k=1-M}^{\infty} \left(\frac{\alpha}{\beta}\right)^k e^{-\alpha\beta} \mathrm{I}_k(\alpha\beta) \tag{110}$$

and

$$1 - Q_M(\alpha, \beta) = e^{-\frac{(\alpha-\beta)^2}{2}} \sum_{k=M}^{\infty} \left(\frac{\beta}{\alpha}\right)^k e^{-\alpha\beta} \mathrm{I}_k(\alpha\beta). \tag{111}$$

Obviously the performance of this approach is entirely dependent on the Bessel function routines. The routines used by `octave` are those written by Amos ([8]) and found on netlib `http:`

//www.netlib.org/amos. These routines are fast and reasonably accurate, especially for small arguments. The worst-case relative accuracy is about ten significant figures, however, so better results should not normally be expected for our situation.

Figures 9, 10, 11, and 12 show the iteration counts for parameters shown. Figures 13, 14, 15, and 16 show the corresponding execution times.

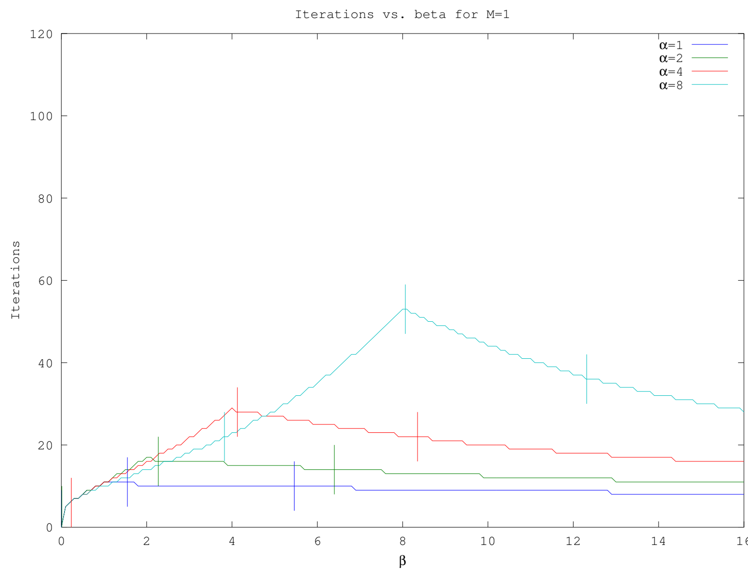

Figure 9: Iteration Count Using Series Method for $M = 1$

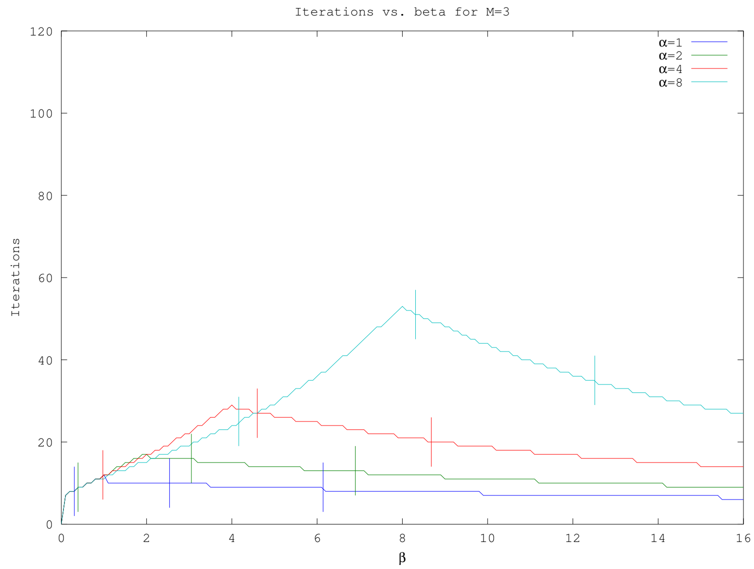

Figure 10: Iteration Count Using Series Method for $M = 3$

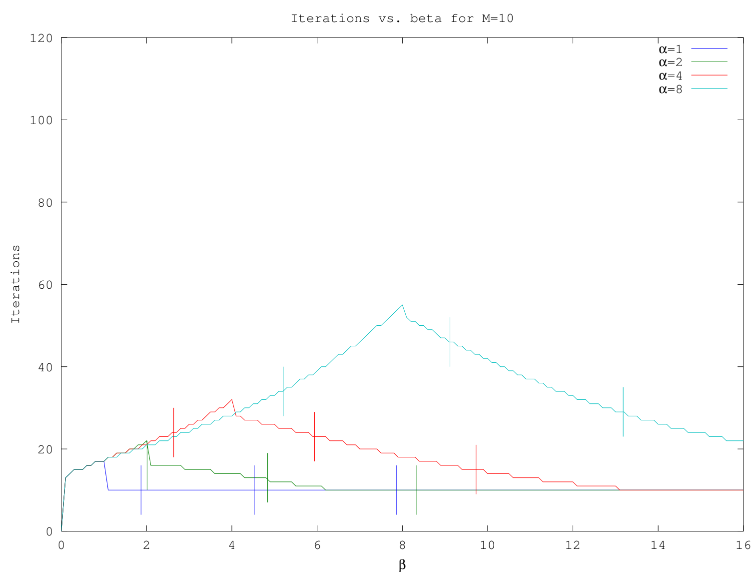

Figure 11: Iteration Count Using Series Method for $M = 10$

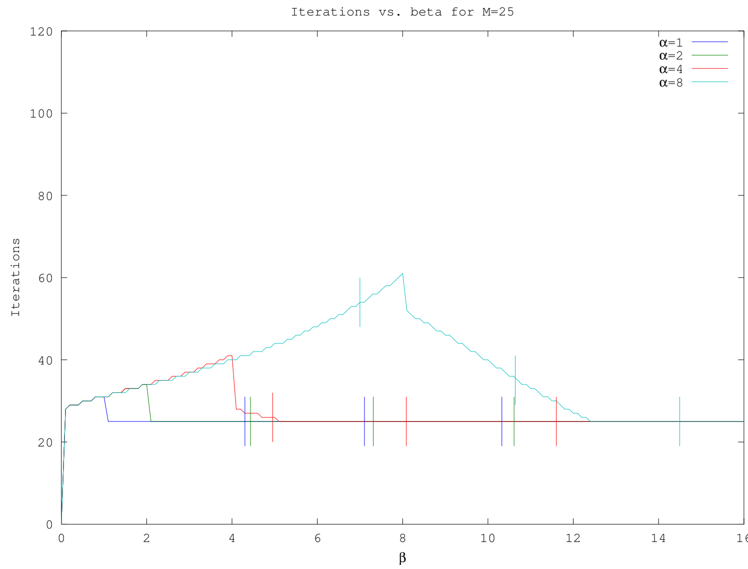

Figure 12: Iteration Count Using Series Method for $M = 25$

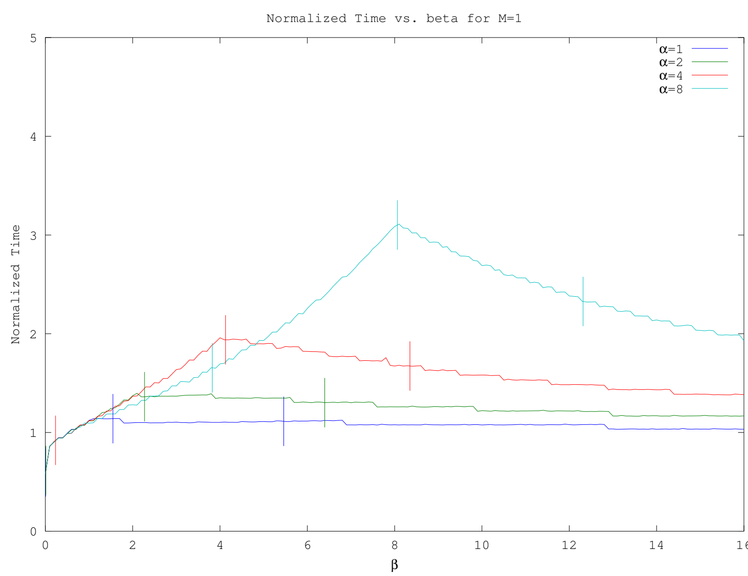

Figure 13: Execution Times Using Series Method for $M = 1$

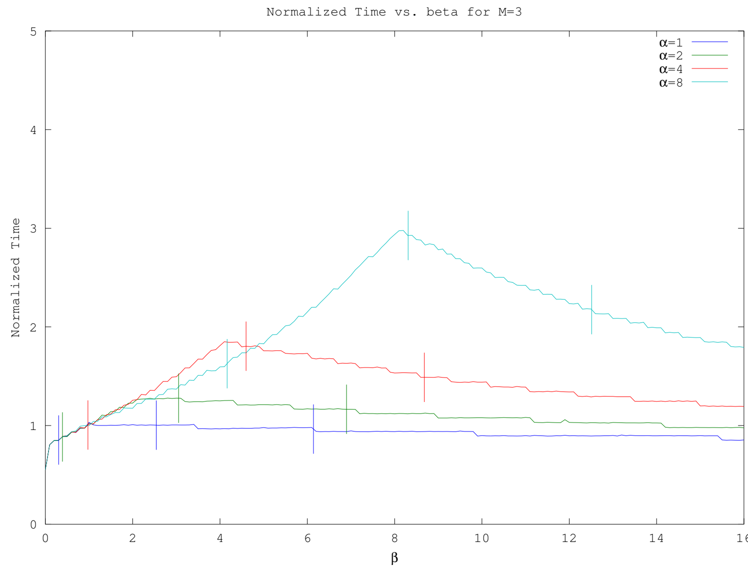

Figure 14: Execution Times Using Series Method for $M = 3$

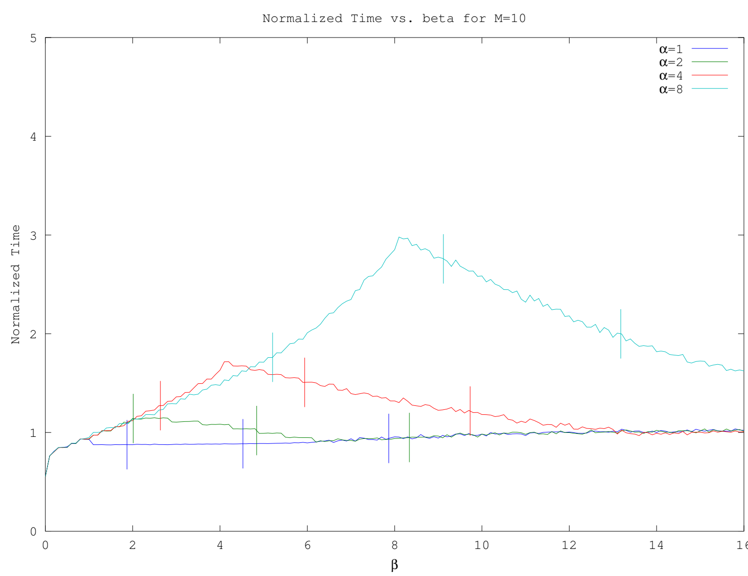

Figure 15: Execution Times Using Series Method for $M = 10$

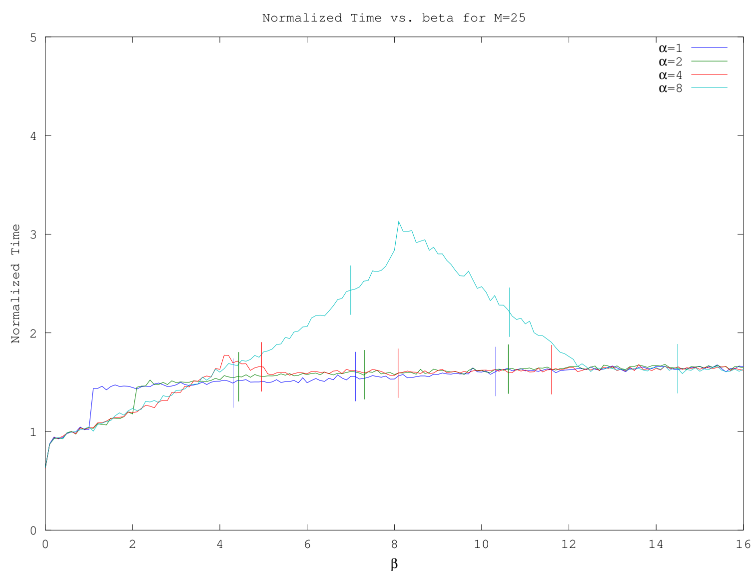

Figure 16: Execution Times Using Series Method for $M = 25$

6.2 Parl's Method

Figures 17, 18, 19, and 20 show the iteration counts for parameters shown. Figures 21, 22, 23, and 24 show the corresponding execution times.

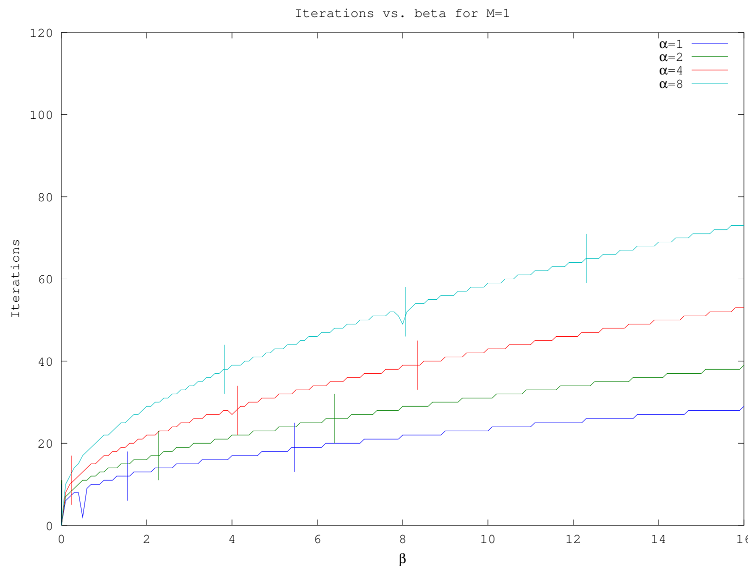

Figure 17: Iteration Count Using Parl's Method for $M = 1$

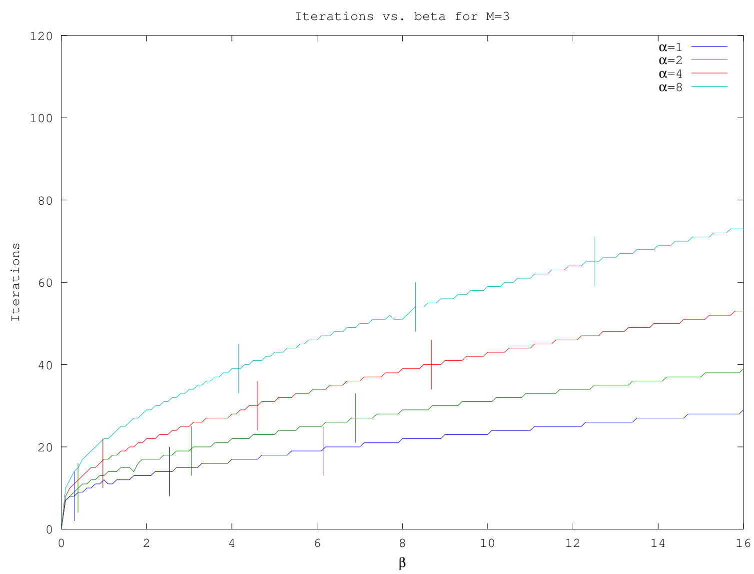

Figure 18: Iteration Count Using Parl's Method for $M = 3$

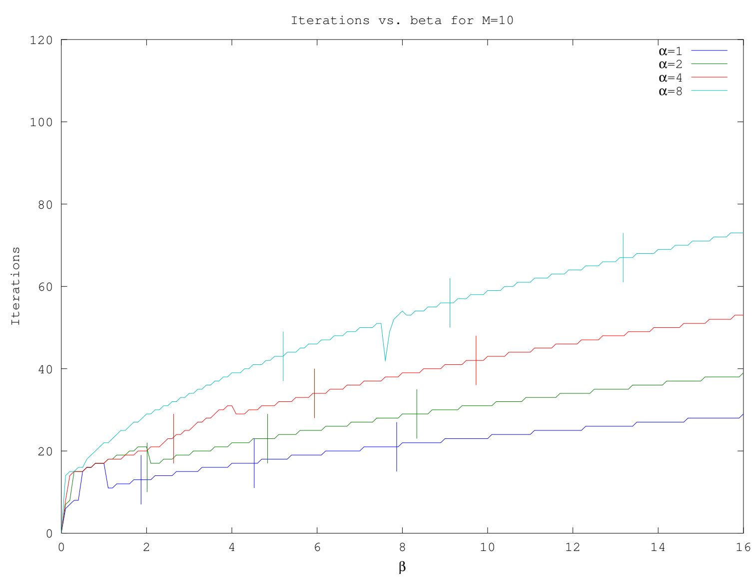

Figure 19: Iteration Count Using Parl's Method for $M = 10$

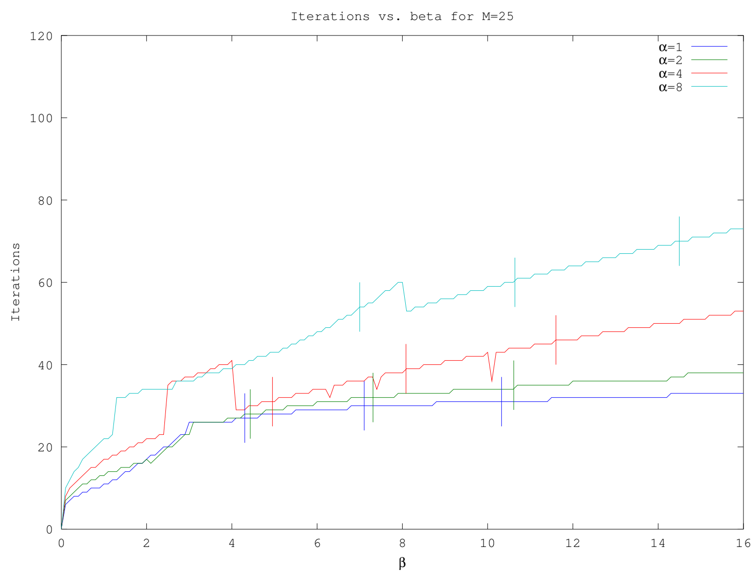

Figure 20: Iteration Count Using Parl's Method for $M = 25$

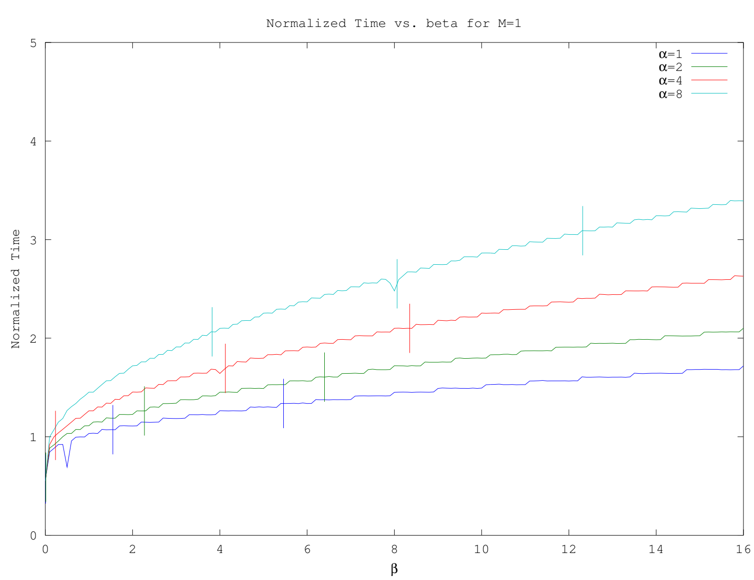

Figure 21: Execution Times Using Parl's Method for $M = 1$

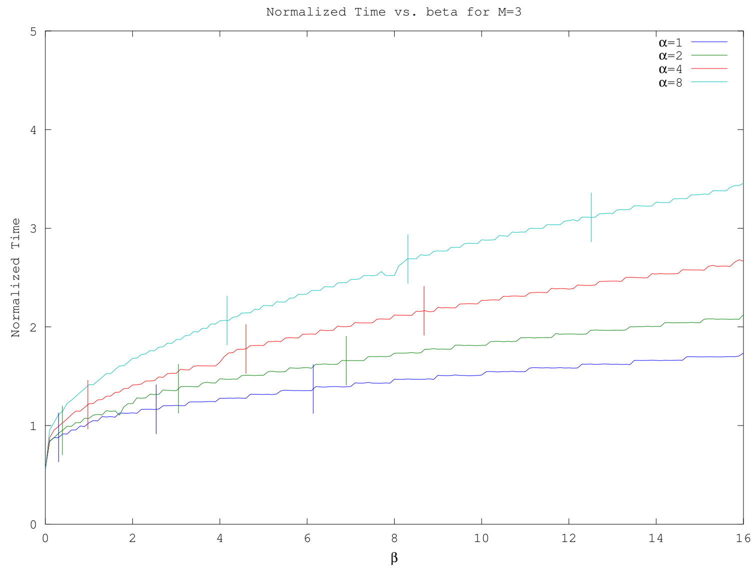

Figure 22: Execution Times Using Parl's Method for $M = 3$

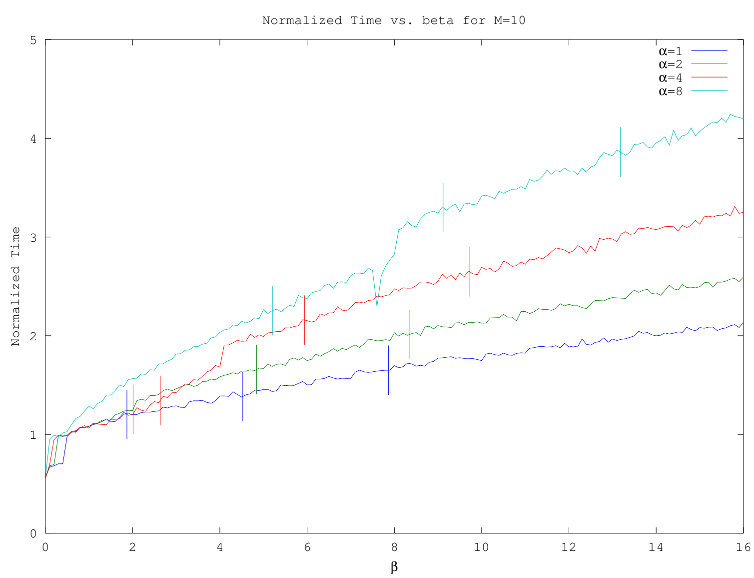

Figure 23: Execution Times Using Parl's Method for $M = 10$

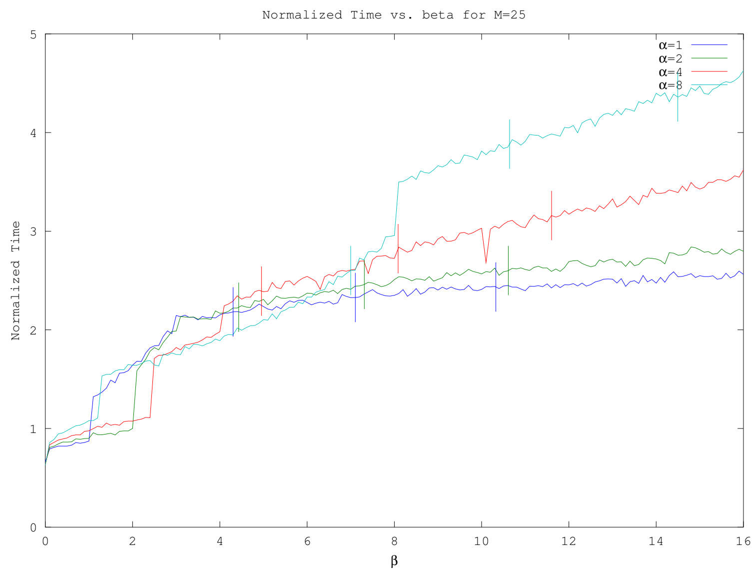

Figure 24: Execution Times Using Parl's Method for $M = 25$

6.3 Dillard's Method

Figures 25, 26, 27, and 28 show the iteration counts for parameters shown. Figures 29, 30, 31, and 32 show the corresponding execution times.

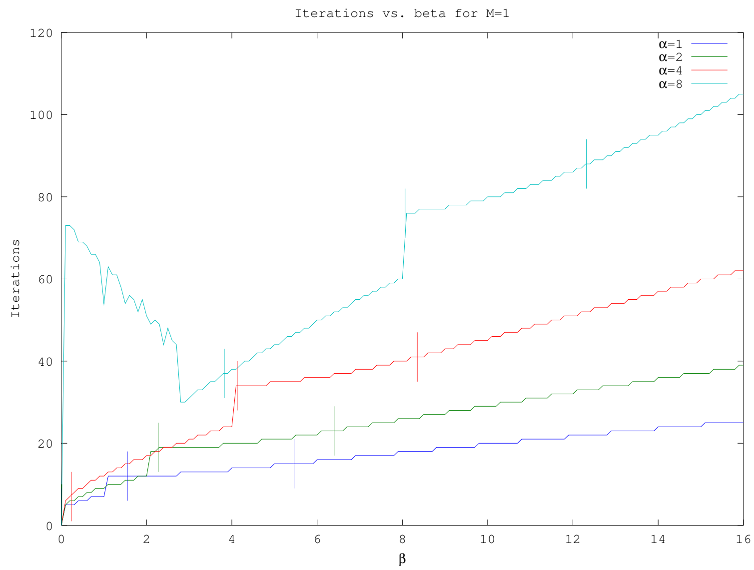

Figure 25: Iteration Count Using Dillard's Method for $M = 1$

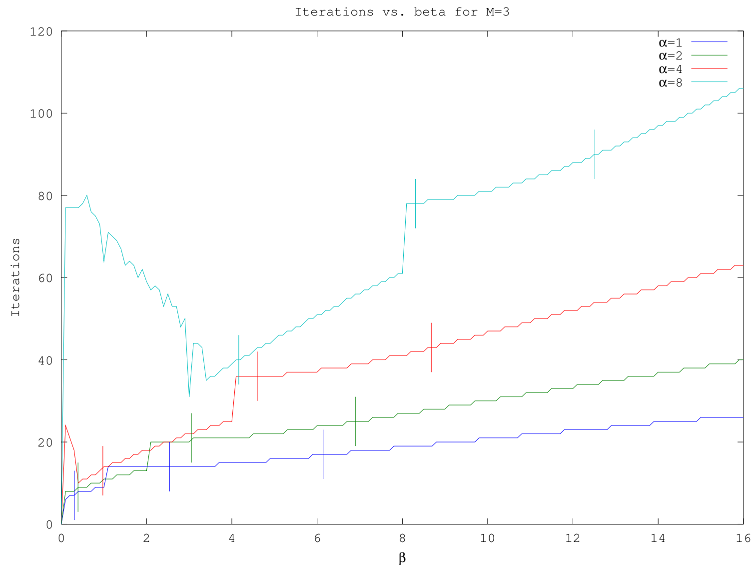

Figure 26: Iteration Count Using Dillard's Method for $M = 3$

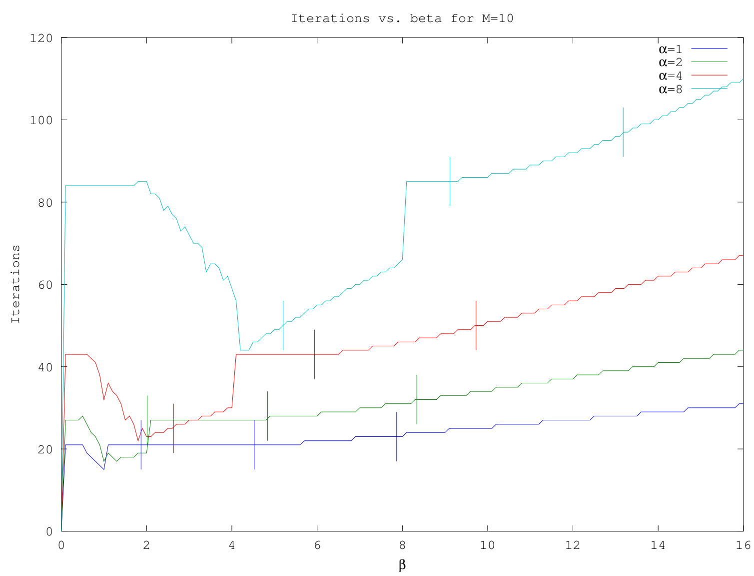

Figure 27: Iteration Count Using Dillard's Method for $M = 10$

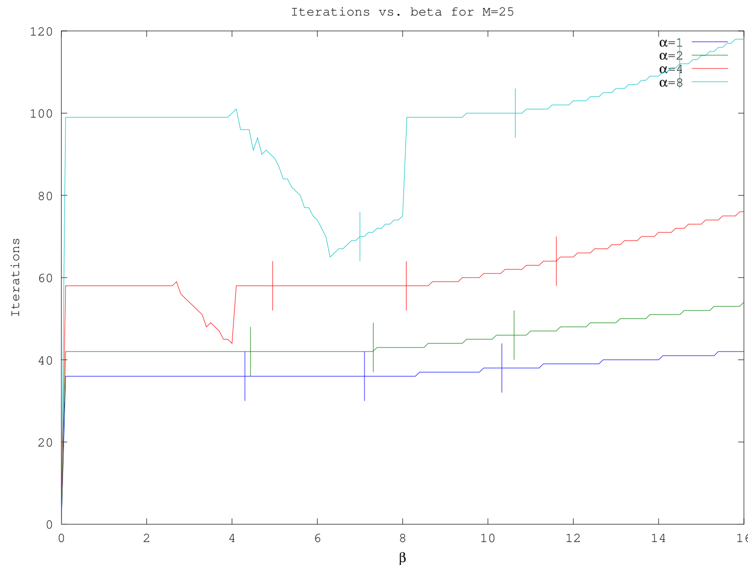

Figure 28: Iteration Count Using Dillard's Method for $M = 25$

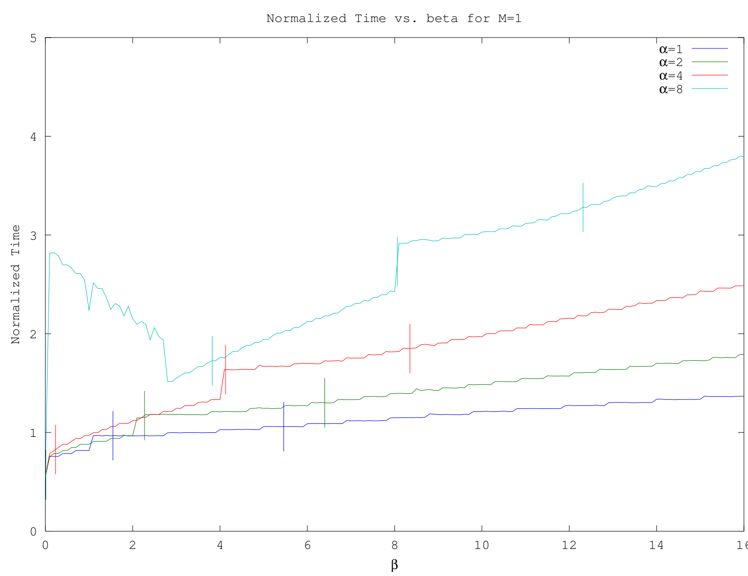

Figure 29: Execution Times Using Dillard's Method for $M = 1$

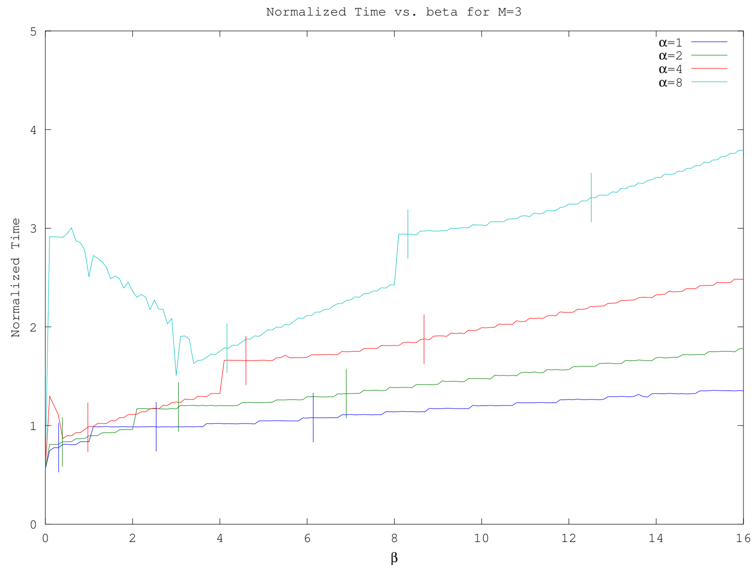

Figure 30: Execution Times Using Dillard's Method for $M = 3$

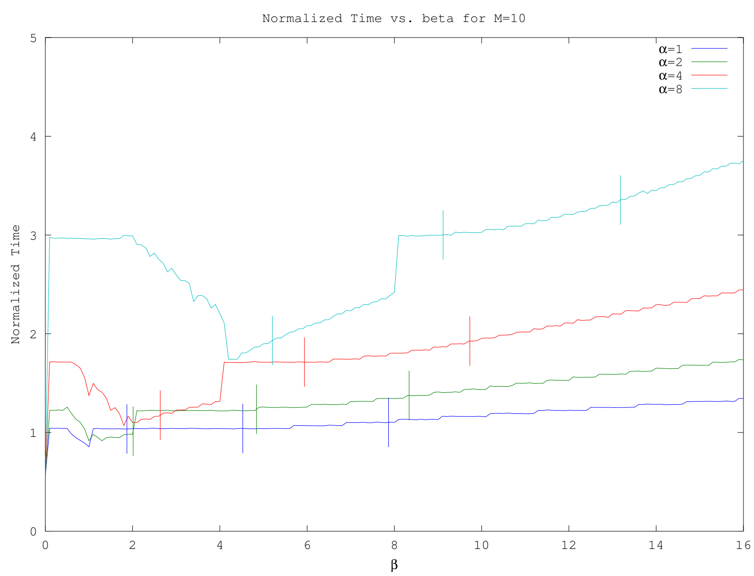

Figure 31: Execution Times Using Dillard's Method for $M = 10$

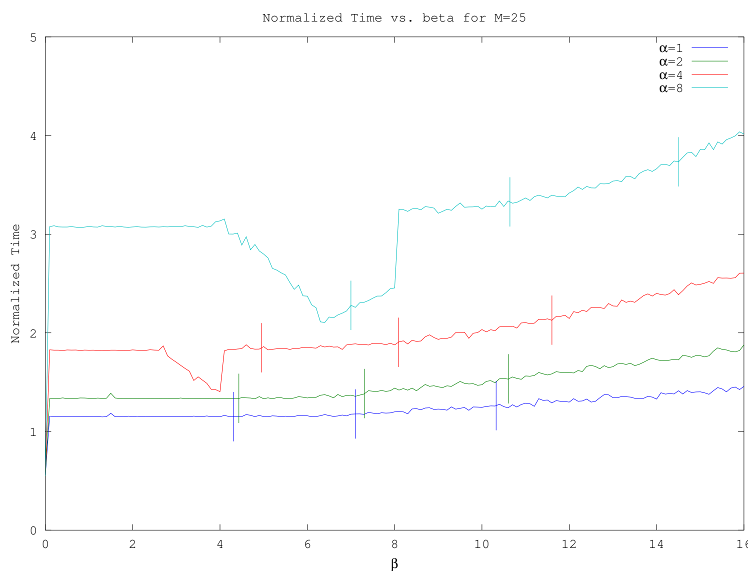

Figure 32: Execution Times Using Dillard's Method for $M = 25$

A Properties of Bessel Functions

Bessel functions are frequently useful when mucking about with probability functions. Here are a few properties from [9]. We will use n to represent integer orders and ν to represent arbitrary orders. The argument z is any complex number.

1. $$\mathrm{I}_n(z) = \mathrm{I}_{-n}(z), \tag{112}$$

2. $$\mathrm{I}_{\nu-1}(z) = \mathrm{I}_{\nu+1}(z) + \frac{2\nu}{x}\mathrm{I}_\nu(z), \tag{113}$$

3. $$1 = \mathrm{I}_0(z) - 2\sum_{n=1}^{\infty} \mathrm{I}_{2n}(z), \tag{114}$$

4. $$e^z = \mathrm{I}_0(z) + 2\sum_{n=1}^{\infty} \mathrm{I}_n(z), \tag{115}$$

5. $$\frac{d^k}{dz^k}\left[z^\nu \mathrm{I}_\nu(\alpha z)\right] = \alpha^k z^\nu \mathrm{I}_{\nu-k}(\alpha z). \tag{116}$$

6. $$\frac{d^k}{dz^k}\left[z^{-\nu} \mathrm{I}_\nu(\alpha z)\right] = \alpha^k z^{-\nu} \mathrm{I}_{\nu+k}(\alpha z). \tag{117}$$

7. $$\mathrm{I}_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta e^{z\cos\theta} d\theta. \tag{118}$$

8. $$\lim_{\alpha\to 0} \frac{\mathrm{I}_\nu(\alpha z)}{\alpha^\nu} = \lim_{\alpha\to 0} \frac{e^{-\alpha z}\mathrm{I}_\nu(\alpha z)}{\alpha^\nu} = \frac{z^\nu}{2^\nu \Gamma(\nu+1)}. \tag{119}$$

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